Building blocks: The formation of extractive structures in networks

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Outline

Overview

- We identify non-competitive network structures (barriers to competition) in which individuals and groups of agents can disrupt trade and information flows between others in the network.
- These disruptive structures relate to a notion of competition on networks—referred to as "contestability". From this, we provide a measurement of power in terms of the brokerage of agents.
- A non–cooperative, strategic form game on a network is developed in which players maximise their brokerage by forming structures and exploiting positions that prevent contestation.
- We look at incomplete, non–empty networks that lack contestation and lend themselves to exploitive agents and the formation of disruptive structures.

Relevant literature (1)

- A *middleman* is a node that controls all pathways from at least one node to at least one other.

- These critical nodes have had a resurgence of attention:
  1. **In economics.** Kalai et al. (1978); Rubinstein and Wolinsky (1987); Biglaiser (1993); Biglaiser and Friedman (1994); Jackson and Wolinsky (1996); Gilles et al. (2006); Masters (2007; 2008); Blume et al. (2007); Goyal and Vega-Redondo (2007); Easley and Kleinberg (2010); Gilles and Diamantarais (2013); and Sims and Gilles (2014).
  2. **In sociology.** Emerson (1962); Granovetter (1973; 2005); Emerson and Cook (1978); Gould and Fernandez (1987); Burt (1992; 2004; 2010); Spiro et al. (2013).

- Middlemen can provide access to new markets, resources, social groups, and opportunities through weak ties. However, due to their position, they can be highly exploitive: rent-seekers, transmission controllers, information brokers.

Relevant literature (2)

- A *block* is a set of at least two nodes that collectively perform a middleman function. Their emergence and relationship to competition on networks has had no attention despite their ability to emerge in almost all non-trivial networks.
- In economic terms, middlemen are equivalent to *monopolists* and blocks are equivalent to *cartels* ; both profit due to the lack of competition regarding their activity in the economy.
- Through these interlinked concepts we analyse the dynamic nature of competition on networks in which agents form structures to exploit collective positions of power.

Preliminaries : Networks and walks

- A **(directed) network** is a pair (N, D) where $N = \{1, 2, \ldots, n\}$ is a finite set of nodes and $D \subset \{(i, j) \mid i, j \in N \text{ and } i \neq j\}$ is a set of **arcs**, being directed relationships from one node to another, where an arc from node i to j is denoted as $ij = (i, j)$.

- An **(i, j)-walk**, as a directed walk on network D, is a tuple of connected nodes $W_{ij}(D) = [i_1, ..., i_m] \subset N$ with $m \geqslant 3$, $i_1 = i$, $i_m = j$, and $i_k i_{k+1} \in D$ for every $k = 1, ..., m-1$.

- There can exist multiple distinct walks from i to j in D. We denote $W_{ij}^{v}(D)$ as the v^{th} distinct walk from i to j in D.

- The class $\mathcal{W}_{ij}(D) = \{W_{ij}^{1}(D), \ldots, W_{ij}^{V}(D)\}$ is a set of sets that consists of all distinct walks from i to j in D, where V is the number of distinct walks. If $V = 0$, then $\mathcal{W}_{ij}(D) = \varnothing$.

Preliminaries : Successors, predecessors, and node deletion

- We use $P_i(D) = \{j \in N \mid \mathcal{W}_{ji}(D) \neq \varnothing \text{ where } i \neq j\}$ to denote i's **predecessor set** and $S_i(D) = \{j \in N \mid \mathcal{W}_{ij}(D) \neq \varnothing \text{ where } i \neq j\}$ to denote i's **successor set**.
- We introduce the **reach** of a node by a modified predecessor set: $\overline{P}_i(D) = P_i(D) \cup \{i\}$.
- Let $D - B$ represent the restricted network obtained by deleting the node set $B \subset N$ from the network D. This is equivalent to:

$$D - B = \{(j, h) \in D \mid j, h \in N \setminus B\}.$$

Defining Middlemen and Blocks (1)

Definition (Middlemen)

Let D be a network on node set N where $i, j, h \in N$.

(a) Node h is an **(i, j)–middleman** if, for some $i, j \in N$ where $\mathcal{W}_{ij}(D) \neq \varnothing$ and $i \neq j$, it holds that:

$$h \in \bigcap \mathcal{W}_{ij}(D) = W^1_{ij}(D) \cap \cdots \cap W^V_{ij}(D),$$

where there exist $V \geqslant 1$ distinct walks from i to j.

(b) The **middleman set** in network D is the collection of all middlemen:

$$\mathcal{M}(D) = \{\, h \mid h \text{ is an (i,j)–middleman for some } i, j \in N \,\}.$$

(c) If $h \notin \mathcal{M}(D)$ then h is a **non–middleman**.

Defining Middlemen and Blocks (2)

Definition (Blocks)

(a) Node set $B_{ij} \subset N$ is an **(i, j)–block** if $\#B \geqslant 2$ and it holds that $\mathcal{W}_{ij}(D) \neq \varnothing$ and $\mathcal{W}_{ij}(D - B) = \varnothing$ for some $i, j \in N$ where $i \neq j$ and $i, j \notin B$.

(b) The **block set** of D is the set of all blocks:

$$\mathcal{B}(D) = \{\, B \mid B \subset N \text{ is an (i, j)–block for some } i, j \in N \,\} .$$

(c) The block set of node h is the collection of blocks that it is a member of, given as:

$$\mathcal{B}_h(D) = \{\, B \in \mathcal{B}(D) \mid h \in B \,\} .$$

(d) The **critical set** of the network D is given as:

$$\mathcal{B}^{\star}(D) = \mathcal{B}(D) \cup \mathcal{M}(D).$$

Properties of Middlemen and Blocks

Proposition

(i) *Every* $i \in \mathcal{M}(D)$ *is an intermediary in* D.

(ii) *A complete network cannot have middlemen or blocks.*

(iii) *Every* $i \in \mathcal{M}(D)$ *has a local clustering co-efficient of less than* 1.

(iv) *If* D *is undirected in that* $(i, j) \in D \iff (j, i) \in D$, *then* $\mathcal{M}_{ij}(D) = \mathcal{M}_{ji}(D)$ *and* $\mathcal{B}_{ij}(D) = \mathcal{B}_{ji}(D) \, \forall \, i, j \in N$.

Theorem

$\mathcal{B}^{\star}(D) \neq \varnothing \iff \exists$ *at least one pair* $i, j \in N$, *where* $i \neq j$, *with* $\min \{\# W_{ij} | W_{ij} \in \mathcal{W}_{ij}\} \geqslant 3$.

Contestability (1)

- Network competition: A set of nodes are *fully contested* if the contesting nodes can perform all the activities of the initial set if the initial set is removed from the network.

- Further, a set of nodes are *partially contested* by the contesting nodes perform some, but not all, of the activities of the initial set if it is removed from the network.

- The **coverage** of node $i \in N$ is given by $P_i(D) \times S_i(D)$. By extension, let $B \subset N$ where $P_B(D) = \bigcup_{i \in B} (P_i(D) \setminus B)$ and $S_B(D) = \bigcup_{i \in B} (S_i(D) \setminus B)$, the coverage of node set B is $P_B(D) \times S_B(D)$.

  A node set is fully contested by another if its coverage is covered by the contesting node set given the removal of the initial node set.

Contestability (2)

Definition (Contestability)

Let D be a network on node set $N = \{1, \ldots, n\}$ where $B, C \subset N$ and $B \cap C = \varnothing$.

(a) Node set B is **fully contested** by C if it holds that:

$$P_B(D) \times S_B(D) \subseteq \bigcup_{j \in C} \left(\overline{P}_j(D - B) \times S_j(D - B) \right).$$

(b) Node set B is **partially contested** by C if it is not fully contested and it holds that:

$$[P_B(D) \times S_B(D)] \cap \bigcup_{j \in C} \left[\left(\overline{P}_j(D - B) \times S_j(D - B) \right) \right] \neq \varnothing.$$

(c) A node set is **uncontested** if it is neither fully nor partially contested.

Contestability (3)

Theorem (Duality)

Let D be a network on node set N.

(i) All middlemen and blocks are not fully contested.

(ii) If node set $K \subset N$ is not fully contested then it is a middlemen if $\#K = 1$ or a block otherwise.

Middlemen and blocks can be partially contested.

Proposition

(i) Sources have no coverage but have the ability to contest other nodes due to their reach.

(ii) Let $B \subset N$ be a block. B must contain all nodes that either fully or partially contest each other for at least one $(i, j) \in P_B(D) \times S_B(D)$.

Redundancy in blocks (1)

- The number of blocks increases with the number of structural holes.
- However, not all of the blocks are equally compelling, there can exist blocks that are *redundant*.

Definition (Redundancy)

Let D be a network on node set $N = \{1, \ldots, n\}$ where $B \subset N$ is a block and $i, j \in N$.

(a) The **brokerage set** of node set $B \subset N$ in the network D, denoted by $\mathcal{Z}_B(D)$, contains all pairs $(i,j) \in P_B(D) \times S_B(D)$ where $\mathcal{W}_{ij}(D) \neq \varnothing$ and $\mathcal{W}_{ij}(D - B) = \varnothing$.

(b) Block B is **redundant** if $\exists\, B' \subset B$ where $\mathcal{Z}_{B'}(D) \supseteq \mathcal{Z}_B(D)$, and **non–redundant** otherwise.

Redundancy in blocks (2)

Proposition

Let D be a network on node set N where $B \subset N$.

- (i) *Any block containing a source and / or a sink is redundant.*
- (ii) $\mathcal{B}_i(D) = \varnothing$ *when* $\mathcal{B}(D) \neq \varnothing$ *if* $i \in \bigcap_{B \in \mathcal{B}^\star(D)} \mathcal{Z}(B)$.
- (iii) *Let node $h \in N$ be uncontested. If $h \in B$ where $B \in \mathcal{B}(D)$ then block B is redundant.*
- (iv) *Let node sets B and B' be blocks. $L = \{B \cup B'\}$ is not a block if and only if $\mathcal{Z}(B) \subseteq B'$ and $\mathcal{Z}(B') \subseteq B$.*
- (v) *Let $B' \subset B$ where $B, B' \in \mathcal{B}(D)$. If $\mathcal{Z}(B') \supseteq \mathcal{Z}(B)$, then no members of the set difference, $B \setminus B'$, neither fully nor partially contests any member of B'.*

Network power

- The unique connectivity of a middleman or block is measured in terms of its brokerage in the network.

Definition (Brokerage)

Let D be a network on node set $N = \{1, \ldots, n\}$ where $B \subset N$. The **brokerage** of node set B is given as:

$$b_B(D) = \sum_{i \in N \setminus B} \# \left[S_i(D) \setminus B \right] - \sum_{i \in N} \# S_i(D - B).$$

Proposition

The limits of the brokerage are: $0 \leqslant b_B(D) \leqslant (n-1)(n-2)$.

- We use brokerage in the payoff function of the block formation game which expresses an analogy of cartel formation in networks.

Setting up the game (1)

- The block formation game, (A, π, D), is a non-cooperative, strategic form game on the player set $N = \{1, \dots, n\}$ in the network D.
- The action set for every player $i \in N$ is given by:

$$A_i = \mathcal{B}_i(D) \cup \{i\}.$$

  If $a_i = B \in \mathcal{B}_i(D)$ then i signals to all $j \in B$, where $i \neq j$, her willingness to form B. If $a_i = i$ then agent i will only exploit her own position.

- Block $B \in \mathcal{B}(D)$ is formed if and only if $a_j = B \, \forall j \in B$.

Setting up the game (2)

- The payoff function for every $i \in N$ is given by:

$$\pi_i(a) = \gamma_{a_i} \left(\frac{b_{a_i}}{\# a_i} \right) - (\# a_i - 1)\, c,$$

where $c \in \mathbb{R}$ is a cost of sending a signal to all other members of the block, and

$$\gamma_{a_i} = \begin{cases} 1 & \text{if } a_j = a_i \, \forall \, j \in a_i \\ 0 & \text{otherwise.} \end{cases}$$

- The payoff function assumes an egalitarian distribution of the brokerage of any block that is formed among all members of that block. Moreover, due to γ_{a_i}, the payoff of i can be dependent on others.
- If $a_i = i$ then $\pi_i(a) = b_i$.
- If $\exists j \in a_i$ where $a_j \neq a_i$ then $\pi_i(a) = -(\# a_i - 1)\, c$.

Equilibrium analysis : Ranking (1)

- Blocks and middlemen are ranked by their *maximal payoff*, given by:

$$\sigma(B) = \frac{b_B}{\#B} - (\#B - 1)c, \text{ for } B \in \mathcal{B}(D) \cup \mathcal{M}(D).$$

  σ ranks $\mathcal{B}(D) \cup \mathcal{M}(D) = \mathcal{B}^\star(D)$.

- Let $\mathcal{B}^\circ(D) = \{B \mid B \in \mathcal{B}^\star(D) \text{ and } \sigma(B) > 0\}$.

- Construct $\mathcal{B} \subseteq \mathcal{B}^\circ(D)$ as follows:

  (1) $B^1 \in \arg\max \{\sigma(B) \mid B \in \mathcal{B}^\star(D)\}$.

  (2) Let $B^1, \ldots, B^m$ be selected. Choose:

$$B^{max} \in \arg\max \left\{ \sigma(B) \,\middle|\, B \in \mathcal{B}^\star(D), B \subset N \setminus \bigcup_{k=1}^{m} B^k \right\}.$$

Equilibrium analysis : Ranking (2)

(3) Continue until:

$$\arg\max \left\{ \sigma(B) \,\middle|\, B \in \mathcal{B}^\star(D), B \subset N \setminus \bigcup_{k=1}^{K} B^k \right\} = \varnothing.$$

Where the outcome is $B^1, \ldots, B^K$.

- Define $\tilde{a} \in A$ for $B^1, \ldots, B^K$ by:
  - $\tilde{a}_i = B^m \, \forall \, i \in B^m$, and
  - $\tilde{a}_j = j \, \forall \, j \in N \setminus \bigcup_{k=1}^{K} B^K$

Equilibrium analysis : Strong Nash equilibrium

Theorem (Strong Nash equilibrium)

$\tilde{a}$ is a Strong Nash equilibrium (SNE).

Corollary

(i) $B^1 \in \mathcal{B}$ *is in SNE.*

(ii) $B^2 \in \mathcal{B}$*, where* $\sigma(B^1) > \sigma(B^2)$*, is an SNE* $\iff B^1 \cap B^2 = \varnothing$.

(iii) *All SNE blocks are non-redundant.*

- Block $B \in \mathcal{B}$ does not emerge in SNE if for some $i \in B\ \exists\, a_i \in A_i \setminus B$ where $\sigma(a_i) > \sigma(B)$ and a_i is in SNE.
- There exist multiple SNE if $\exists\, B, B' \in \mathcal{B}$ where $\sigma(B) = \sigma(B')$, $B \cap B' \neq \varnothing$, and $\nexists\, B'' \in \mathcal{B}$ such that $\sigma(B'') > \sigma(B)$, $B \cap B'' \neq \varnothing$, $B' \cap B'' \neq \varnothing$, and B'' is in SNE.

Equilibrium analysis : Nash equilibrium

Theorem (Nash equilibrium)

$B \in \mathcal{B}^\star(D)$ is not in a Nash equilibrium (NE) $\iff \exists\, B' \in \mathcal{B}^\star(D)$ such that $\sigma(B') > \sigma(B)$, $B \cap B' \neq \varnothing$, and $\#B' = 1$.

Corollary

(i) *$B \in \mathcal{B}$ is strictly dominated by $B' \in \mathcal{B}$ if and only if $\sigma(B') > \sigma(B)$, $B \cap B' \neq \varnothing$, and $\#B' = 1$.*

(ii) *If $i \in \mathcal{M}(D)$ is uncontested then all $B \in \mathcal{B}_i(D)$ will not be in NE.*

(iii) *Both redundant and non-redundant blocks form in NE.*

- Equilibrium analysis can be intuitively applied to *Monadic Stability* (Gilles and Sarangi, 2010) as a form of farsighted block formation.

Example 1

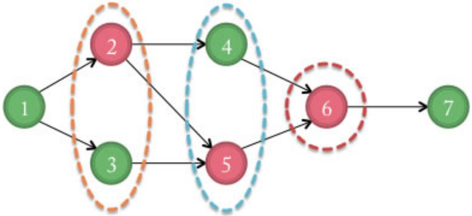

Figure : Network D where $\mathcal{M}(D) = \{2, 5, 6\}$.

- The payoff to all players in the above network, D, *without* the formation of blocks is 8, where : $b_2(D) = 1$, $b_5(D) = 2$, and $b_6 = 5$.

- Unique SNE where blocks $B = \{2, 3\}$ and $B' = \{4, 5\}$ are formed and player 6 exploits her middleman position since she is uncontested. The total payoff is:

$$\sum_{i \in N} \pi_i(\tilde{a}) = 0 + 2 + 2 + 3 + 3 + 5 + 0 = 15.$$

Example 2 (a)

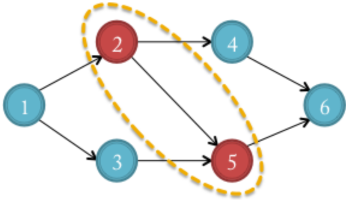

Figure : Network D' where $\mathcal{M}(D') = \{2, 5\}$.

- In network D' player 7 has been removed meaning that player 6 is no longer a middleman.
- Block $B = \{2, 5\}$ is formed in an SNE. Notably, B consists of two middlemen highlighting that middlemen have an incentive to form blocks if they are partially contested by each other.
- The total payoff to block B is 3.

Example 2 (b)

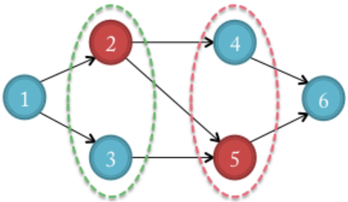

Figure : Network D' highlighting the other SNE.

- Blocks $B = \{2, 3\}$ and $B' = \{4, 5\}$ are formed in the other SNE. In this situation there exist two blocks each containing a middleman and a non–middleman. Note that players 2 and 5 earn a payoff of 1.5 each regardless of the block they participate in.
- The total payoff to all players is 6.

Mass and control (1)

Definition

The **mass** of a network, denoted by $\mathbb{M} \subseteq N$, refers to the set of all nodes that are middlemen or members of stable blocks in all SNE for a given block formation game.

- $\tilde{a}$ corresponds to a SNE in a given block formation game (A, π, D). There exists multiple $\tilde{a}$ if some conditions (noted above) hold.
- Each SNE has a corresponding total payoff: $\pi(\tilde{a}) = \sum_{i \in N} \pi_i(\tilde{a})$. We can note the maximum total payoff by comparing the payoff over all $\tilde{a}$ for a given game:

$$\pi^{MAX} \in \arg\max \left\{ \pi(\tilde{a}) \middle| \pi(\tilde{a}) = \sum_{i \in N} \pi_i(\tilde{a}) \, \forall \, \tilde{a} \text{ in } (A, \pi, D) \right\}$$

Mass and control (2)

- The *control co-efficient* for a given network, D, is given as:

$$\nu(D) = \frac{\pi^{MAX}}{\frac{n}{2}(n-1)(n-2)},$$

where π^{MAX} is the maximum total payoff for the block formation game on the network D and $\nu(D) \in [0,1]$.

- As $\nu(D)$ is closer to 1 there exist more opportunities for blocks to form and middlemen to exploit their position.
    - For an undirected star $\nu(D^\star) = \frac{2}{n}$, and for a directed cycle $\nu(D^\circ) = 1$.

Claim

There exists a positive relationship between the size of the networks mass and the control co-efficient of the network.

Concluding remarks

- We have noted the importance of middlemen and blocks as sets of nodes that have the ability to exploit their position and disrupt the operations in a network due to their lack of contestation.
- Blocks are formed in equilibrium when sets of nodes partially contest each other. Middlemen have the most power in dictating whether blocks are formed or not, therefore dictating the equilibrium.
- Blocks can consist solely of middlemen, solely of non-middlemen, or a mixture of both.
- The mass of a network indicates the potential exploitation on a network and the robustness of the exploitation.