Building blocks: The formation of extractive structures in networks*

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Abstract

The diffusion of information in a network strongly relies on the involvement of so-called critical nodes or "middlemen". In particular, middleman positions provide power to extract rents from mediating control in the network. This paper provides a game theoretic analysis of how groups of individuals attain collective middleman control through cooperation and form a "block" in the network.

We link block formation to a notion of network competition. From this we develop measures of a node's power based on a lack of contestability and their ability to broker relations.

We show that there exist strong Nash equilibria in the block formation process that have appealing normative properties. Furthermore, we investigate belief-based equilibrium concepts that identify alternative blocking patterns.

Keywords: critical nodes, network analysis, contestability, blocks

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1 Introduction

Many social and economic phenomenon can be intuitively explained in terms of networks. Within networks agents are endowed with positional, and thus *relational*, attributes dependent on the other agents that they are directly and indirectly connected to and also the order of those connections. This relational context adds a new dimension to typical economic analysis. Increasingly, network analysis is being used to investigate social and economic processes including trade (Kranton and Minehart, 2001; Blume, Easley, Kleinberg, and Tardos, 2009), bargaining (Polanski, 2007), competition between agents (Sims and Gilles, 2014) and many other phenomena, with insightful results.

This article provides an elaboration of the process of competition in networks focussing on coalitions of nodes—termed as *blocks*—that collectively attain a non–competitive position in a network. Blocks use their non–competitive position to exploit indirect relationships between other agents in the network that require at least one member of the block to be functional. Despite our main contextual reference being exploitation and extraction in trade flows, the analysis itself remains general implying that the network is able to represent many different socio-economic activities such as: investment and loan provision (Gai and Kapadia, 2010; Jackson, Elliot, and Golub, 2014), shareholdings and corporate ownership (Vitali, Glattfelder, and Battiston, 2011), learning and information dissemination (Golub and Jackson, 2010), advice and influence (Krackhardt, 1987), or favour provision (Jackson, Rodriguez-Barraquer, and Tan, 2012), etc. Further still, we use directed networks for generality as any insights are equally applied to strongly connected undirected networks.

1.1 Exploitation and power in networks

The work here broadly applies the notions of critical nodes and node cut sets, originally developed in graph theory, to situations in economics and sociology. In social and economic terms, the graph theoretic notion of a critical node is analogous to that of a *middleman*. There exists much work assessing the importance of critical nodes in economics. We specifically note seminal work developed by Kalai, Postlewaite, and Roberts (1978) which was extended by Jackson and Wolinsky (1996) and further elaborated upon by Gilles, Chakrabarti, Sarangi, and Badasyan (2006) with respect to undirected networks. This literature showed that middleman positions are important in networked intermediation and that such critical nodes can extract significant gains from their positions within a cooperative game theoretic framework. Indeed, the insights were analogous to those found by Rubinstein and Wolinsky (1987). There also exists much work in sociology regarding critical nodes and power. Emerson (1962) illustrates a theory of power relations by assessing the dependence of each player on the other, and thus the number of alternatives that each player has at their disposal for the achievement of a given task. The notions of power–dependence relations were extended to analyse the power of agents in exchange networks by Cook, Emerson, Gillmore, and Yamagishi (1983). Gould and Fernandez (1989) note that there can exist multiple types of broker, which acts as a critical node, and provide quantitative measurements for these nodes primarily based on the notion of betweenness centrality. Gould (1989) builds on these early insights, developing a measure for an agents inter–clique brokerage. More recent research has investigated more dynamics of brokerage (Spiro, Acton, and Butts, 2013). Despite the vigorous research from both fields into the interlinked notions of middlemen, brokerage, and critical nodes, he notion of node cut sets still require intuitive application to scenarios in both sociology and economics. This article provides one application of cut sets in the form of block formation.

Exploitation and power is closely related to the notion of *competition* in economic systems. Competition has been at the heart of market theory and traditional economics since the formal introduction of Bertrand competition, which claims that if there exists two or more producers for some homogeneous product in a given market then the producers will continually reduce their price levels so that all producers are selling their outputs at the marginal cost of production. The assumption of competitive systems is at the heart of both micro and macroeconomic modelling, however there does not yet exist a broad range of literature regarding competition in networked markets.

Easley and Kleinberg (2010) provide a baseline model regarding three classes of players exchanging with each other in a tripartite network subsequently highlighting potential notions of perfect competition and monopolisation. These competitive notions are built from a combination of Bertrand competition and more implicitly from insights of power introduced by Emerson (1962). Gilles and Diamantaras (2013) note the importance of middlemen, or *platforms*, regarding their ability to extract rents from its users that use the platform to interact. Sims and Gilles (2014) provide a formal definition of both strong and weak middlemen in directed networks and in doing so turn applying this to a network–centric notion of competition which is effectively a generalisation of Easley and Kleinberg's notion of competition.

Goyal and Vega-Redondo (2007) provides an elaborate example of network formation with surplus–generating economic exchange. The surplus from exchange is split evenly between players that are directly connected and any indirect exchange is split with the set of intermediaries. Players that span structural holes, á la Burt (1992), can therefore be highly extractive depending on the players whose exchange they intermediate.

To this point there is a serious deficiency regarding groups of agents collectively attaining extractive positions in networks, comparable to the notion of a *cartel* in economic theory. We partially alleviate this problem noting that the formation of blocks in networks is analogous to the formation of cartels in market economies. However, when applied to networks we find that there can emerge situations where monopolists have an incentive to form a block with other monopolists as well as other powerless players.

1.2 Middlemen as entrepreneurs

The notion of entrepreneurship in networks was initially pioneered by Burt, who suggested that nodes who possess bridge relations (Granovetter, 1973) which span multiple components are able to filter diverse information between multiple, otherwise disconnected, components and cliques. In this way entrepreneurs can bring players together, keep people apart, and generally have better ideas that are subsequently evaluated by a larger, more diversified group of peers. This has an easy translation from sociology to economics. Indeed, Sims and Gilles (2014) show that middlemen are nodes that have a unique connection in a network, meaning that their removal impacts the connectivity of other players in the network. Middlemen are therefore comparable to entrepreneurs: they posses some connectivity that allows them to indirectly connect a set of nodes that would not have been connected otherwise. Middlemen therefore perform a role that is unique to all other players in the network much like the typical definition of an entrepreneur under the more classical definitions provided by Schumpeter who suggests that, "the function of the entrepreneur is to reform or revolutionise the pattern of production"(Schumpeter, 1942, p. 84). This reform or revolution is achieved through multiple potential actions, all of which correspond to the creation of some form of novelty that has a *disequilibriating* force on the economy, transiting it from one state of affairs to another through some process of "Creative Destruction". Indeed, the entrepreneur creates a role that is unique and has a substantial impact to the structure of production and trade.

The formation of blocks can also be treated as an entrepreneurial act. We find that groups of players can operate in a coordinated manner to form structures that are exploitive to at least one indirect relationship. The formation of a block therefore changes the context of trade as the set of nodes subsequently performs a connection that cannot be contested by other nodes in the network.

Outline of Article. Section 2 provides background definitions and properties of networks, middlemen, and blocks. We relate these positions in networks to notions of contestability and block redundancy. Section 3 provides an analysis of a general block formation game, multiple equilibrium solutions are assessed. Section 4 applies the analysis to the case of brokerage. Finally, section 5 concludes.

2 Networks, Middlemen and Blocks

2.1 Networks

We consider directed networks defined as a pair (N, D) where $N = \{1, 2, \dots, n\}$ is a finite set of *nodes* and $D \subseteq \{(i, j) \mid i, j \in N \text{ and } i \neq j\}$ is a set of *arcs*, being directed relationships from one node to another. Each node represents a self-motivated decision maker. An arc from node i to j is denoted as $ij = (i, j)$ which is distinct from $ji = (j, i)$. We denote a directed network (N, D) by D unless N is ambiguous [1].

A *walk* from i to j in a directed network D—or an (i, j)−*walk*—is a set of connected nodes $W_{ij}(D) = \{i_1, \dots, i_m\} \subset N$ with $m \geq 2$, $i_1 = i$, $i_m = j$, and $i_k i_{k+1} \in D$ for all $k = 1, \dots, m-1$. In many cases there are multiple walks from i to j in a directed network D. Therefore, we denote $W^v_{ij}(D)$ as the v^{th} distinct walk from i to j in D. The class $\mathcal{W}_{ij}(D) = \left\{W^1_{ij}(D), \dots, W^V_{ij}(D)\right\}$ consists of all distinct walks from i to j in D, where $V = \#\mathcal{W}_{ij}(D)$ is the total number of distinct walks. If $V = 0$, then $\mathcal{W}_{ij}(D) = \varnothing$.

The directed network D is (weakly) *connected* if $\mathcal{W}_{ij}(D) \neq \varnothing$ and/or $\mathcal{W}_{ji}(D) \neq \varnothing$ for all nodes $i, j \in N$. The network D is *strongly connected* if $\mathcal{W}_{ij}(D) \neq \varnothing$ as well as $\mathcal{W}_{ji}(D) \neq \varnothing$ for all nodes $i, j \in N$. Clearly, strongly connected networks are always connected and connected undirected networks are necessarily strongly connected.

If $\mathcal{W}_{ij}(D) \neq \varnothing$ then j is the *successor* of i and i is the *predecessor* of j in D. We denote $S_i(D) = \{j \in N \mid \mathcal{W}_{ij}(D) \neq \varnothing\}$ as i's *successor set*, where $i \notin S_i(D)$. We let $\overline{S}_i(D) = S_i(D) \cup \{i\}$ be defined as the *origin* of node i. Likewise, $P_i(D) = \{j \in N \mid \mathcal{W}_{ji}(D) \neq \varnothing\}$ denotes i's *predecessor set*, where $i \notin P_i(D)$. We let $\overline{P}_i(D) = P_i(D) \cup \{i\}$ be defined as the *reach* of node i. Finally, we define $s_i(D) = \{ j \in N \mid (i, j) \in D \}$ as all of the *direct successors* of i in D, and $p_i(D) = \{j \in N \mid (j, i) \in D\}$ as all the *direct predecessors* of i in D.

For a connected network D, the node set N can be partitioned into three disjoint subsets: sources, sinks, and intermediaries. Node i is a *source* if $s_i(D) \neq \varnothing$ and $p_i(D) = \varnothing$; i is a *sink* if $s_i(D) = \varnothing$ and $p_i(D) \neq \varnothing$; and i in an *intermediary* if $s_i(D) \neq \varnothing$ and $p_i(D) \neq \varnothing$.

[1] We remark that the analysis throughout this paper can equally be applied to undirected networks in which all arcs are reciprocated such that $ij \in D$ if and only if $ji \in D$ for all $i, j \in N$.

Finally, for $B \subset N$, let $D - B$ be defined by

$$D - B = D_{N \setminus B} = \{(j,h) \in D \mid j,h \in N \setminus B\} . \tag{1}$$

Therefore, $D - B$ is the restricted network that removes the node set B and all arcs to and from the nodes in B.

2.2 Middlemen and blocks

Power in social and economic networks tends to rest on individuals and groups that have an ability to broker relationships. Sims and Gilles (2014) show that in Renaissance Florence the House of Medici had the most powerful brokerage position with respect to marriage relations between opposing elite political factions, meaning that the Medici were able to exploit their position by influencing activity in that period.

This perception of brokerage power can be extended by furthering the definition of middlemen to encapsulate sets of players that collectively form a middleman position. A set of players that collectively form a middleman position is referred to as a *block*.

Definition 2.1 *Let D be a network on node set N where $i, j, h \in N$ are distinct nodes.*

(a) *Node h is an (i, j)–middleman if it holds that:*

$$h \in \bigcap \mathcal{W}_{ij}(D) \setminus \{i, j\}.$$

Likewise, node set $B \subseteq \bigcup \mathcal{W}_{ij}(D) \setminus \{i, j\} \subset N$ is an (i, j)–block if it holds that $B \geqslant 2$ and $B \cap W_{ij}(D) \neq \varnothing$ for every $W_{ij}(D) \in \mathcal{W}_{ij}(D)$.

(b) *The **middleman set** of network D is the collection of all middlemen:*

$$\mathcal{M}(D) = \{h \mid h \text{ is an (i,j)–middleman for some } i, j \in N\} .$$

(c) *The **block set** in network D is the set of all blocks:*

$$\mathcal{B}(D) = \{B \mid B \subset N \text{ is an (i,j)–block for some } i, j \in N\} .$$

The block set for some node $i \in N$ is given by all blocks that she is a member of:

$$\mathcal{B}_i(D) = \{ B \in \mathcal{B}(D) \mid i \in B \} .$$

(d) *The **critical set** of the network D is given by*

$$\mathcal{B}^{\star}(D) = \mathcal{M}(D) \cup \mathcal{B}(D).$$

Node set N can therefore be partitioned into two disjoint sets of nodes: middlemen and non–middlemen. A middleman is a singleton node that lies on all walks from at least one node to at least one other, and a block is a node set that fulfils the same function as a middleman. Blocks and middlemen are required for i's indirect interaction with j in an incomplete and non–empty network, meaning that the removal of a block would stop the interaction from i to j.

Definition 2.1 has equal application to both undirected and directed networks. A block may also contain a middleman or multiple middlemen, but that is not necessary. Indeed, in an undirected network a block can be translated into a node cut set, and therefore a middleman is analogous to a singleton node cut set whose removal partitions a network into multiple connected components.

A number of characteristics can be derived from the assessment of middlemen and blocks. We give the next properties without proof.

Properties 2.2 *Let D be a network on node set N and let $i, j \in N$ with $i \neq j$.*

(i) *Middlemen and blocks can be defined in terms of their disconnectivity of the network. Let $h \in \mathcal{M}(D)$ be an (i, j)–middleman, then it must be that $\mathcal{W}_{ij}(D) \neq \varnothing$ and $\mathcal{W}_{ij}(D - B) = \varnothing$ for some $i, j \in N$ where $i \neq j$. Let $B \subset N$ be an (i, j)–block, then it also must be that $\mathcal{W}_{ij}(D) \neq \varnothing$ and $\mathcal{W}_{ij}(D - B) = \varnothing$ for some $i, j \in N$ where $i \neq j$.*

(ii) *There may exist multiple (i, j)–middlemen and (i, j)–blocks.*

(iii) *Let $h \in N$ be an (i, j)–middleman and $h \in B \subset N$. Now B is an (i,j)–block if and only if $i, j \notin B$.*

The following theorem addresses the existence of blocks and middlemen in a network.

Theorem 2.3 *Let D be at least a weakly connected directed network on N. Then $\mathcal{B}^\star(D) \neq \varnothing$ if and only if there exist $i, j \in N$ with $i \neq j$ such that*

$$\min \left\{ \#W_{ij}(D) \,\middle|\, W_{ij}(D) \in \mathcal{W}_{ij}(D) \right\} \geqslant 3. \tag{2}$$

We note that this is an extremely weak requirement for the existence of blocks and \ or middlemen in a network. The corollary below follows directly from Theorem 2.3.

Corollary 2.4 *$\mathcal{B}^\star(D) = \varnothing$ in both empty and complete networks.*

Proof. Theorem 2.3 notes that $\mathcal{B}^\star(D) = \varnothing$ when the maximum geodesic walk from one node to another in the network is less than 3, suggesting that there needs to be indirect intermediation between nodes for blocks and middlemen to emerge. In the case of an empty network, D, on node set N, $\mathcal{W}_{ij}(D) = \varnothing \,\forall i, j \in N$, in which case $\# W_{ij} = 0 \,\forall\, i, j \in N$. In the case of a complete network $p_i(D) = N \setminus \{i\} \,\forall\, i \in N$ and $s_i(D) = N \setminus \{i\} \,\forall\, i \in N$, in which case $\min \left\{ \# W_{ij}(D) \mid W_{ij}(D) \in \mathcal{W}_{ij}(D) \right\} = 2 \,\forall\, i, j \in N$. Therefore a non-trivial critical set exists in all incomplete, non-empty networks with at least 3 nodes that are all connected by a walk only as noted by Theorem 2.3. ■

Following from the proof, $\mathcal{B}(D) \neq \varnothing \iff \exists\,(N, D)$ where $\# N \geqslant 4$ and at least 4 nodes are either directly or indirectly connected together such that $\exists\, i, j \in N$ and $\min \left\{ \# W_{ij}(D) \mid W_{ij}(D) \in \mathcal{W}_{ij}(D) \right\} \geqslant 3$. Therefore, the block set of a network is non-empty in all incomplete, non-empty networks with at least 4 nodes that are all connected by a walk. This is because the formation of a block requires the presence of at least 4 nodes.

2.3 Network Contestability

Blocks and middlemen introduce a topological perspective on competition in networks. Blocks and middlemen are critical to the structure and functioning of a network since their removal leads to both direct and indirect disconnections, propagating a deterioration of the networks' functionality. Sims and Gilles (2014) introduce the notion of network contestability as a descriptor of node-based competition in directed networks. Here we enhance this concept to describe group-based anti-competitive structures in networks.

Nodes represent abilities which are determined by their individual *coverage* in a network, i.e., the relations that they (indirectly) negotiate. Formally, let D be a network on node set N. The *coverage* of node $i \in N$ is given as $\mathrm{Cov}_i(D) = P_i(D) \times S_i(D)$, i.e., all the node pairs that node i intermediates. An *intermediation* of i is now given as an element in its coverage, $(j, k) \in P_i(D) \times S_i(D)$.

By extension, we can define the coverage of arbitrary node sets.

Definition 2.5 *Let $B \subset N$ be some node set in the network D. Then the* ***coverage*** *of B is given by*

$$Cov_B(D) = \bigcup_{i \in B} \left[\, (P_i\,(D) \setminus B) \times (S_i\,(D) \setminus B)\, \right]. \tag{3}$$

Middlemen and blocks have at least one intermediation that is not *contested* by some alternative set of nodes and, therefore, have a monopoly in performing this intermediation. We claim that contestation can take two different forms: (1) A set of nodes $B \subset N$ is *fully contested* if the contesting nodes can perform all of its intermediations iin $D - B$; and (2) A set of nodes B is *partially contested* by a contesting set C if C can perform some, but not all, of the intermediations of B in $D - B$. A more formal definition is provided below in Definition 2.6.

Definition 2.6 *Let D be a network on node set N and let $B, C \subset N$ with $B \cap C = \varnothing$.*

(a) *Node set B is **fully contested** by node set C if it holds that:*

$$Cov_B(D) \subseteq \bigcup_{j \in C} \left(\overline{P}_j(D - B) \times \overline{S}_j(D - B) \right). \tag{4}$$

(b) *Node set B is **partially contested** by node set C if B is not fully contested by C and it holds that for some $j \in C$:*

$$\left[\left(\overline{P}_j(D - B) \times \overline{S}_j(D - B) \right) \right] \cap Cov_B(D) \neq \varnothing. \tag{5}$$

(c) *Node set B is **uncontested** in D if B is neither fully nor partially contested by any other node set in D.*

The distinction between fully contested and partially contested must be made due to the next discussion of block formation. The concept of partial contestability allows us to consider the notion of competition in networks in a deeper way. For example, we note that middlemen and blocks can be partially contested but never fully contested meaning that there can exist *asymmetric contestation* in that node i can (fully) contest j, but j can only partially contest node i.

Contestability is a form of competition in a network. This form of competition refers only to the specific connectivity of nodes in a network as opposed to the actual output of the nodes, which is the main focus of traditional "market competition" discussed in economic market theory. Indeed, in the case of contestability we do not take into consideration the output of each node, or what each node adds to the network. Instead, we only take into consideration the ability of nodes to pass some unchanging output or information through a network. Logically, if each node in the network produces a unique output then the notion of competition breaks down as no node can compete against any others' output, however nodes can still contest each other with respect to Definition 2.6.

Within economic network analysis, competition and contestability should not be perceived as synonyms, but should instead be considered as compliments. The concept of contestability could be extended with respect to the production of outputs on a network in terms of market competition, however here we limit our discussion to network topological matters only.

The definitions of blocks, middlemen, and contestability leads to the following duality seminally discussed in Sims and Gilles (2014).

Theorem 2.7 *Let D be a connected network on node set N.*

(a) *All middlemen and blocks are not fully contested.*

(b) *If node set $B \subset N$ is not fully contested, then B is either a middleman or a block.*

The superset of a block can still be uncontested, but not always so. Consider a connected network D on N with $n \geqslant 3$ that admits a block $B \subset N \setminus \{i\}$ and the node set $C = N \setminus \{i\}$ for some $i \in N$. Then the node set C is not a block and is contested by i irrespective of whether any middlemen are in the node set C or not. More generally, we note that larger node sets do not necessarily equate to blocks and therefore do not necessarily have more brokerage power, even if a subset of the node set is itself a block. We conclude with a number of additional properties regarding contestability.

Properties 2.8 *Let D be a network on node set N.*

(i) *Sources have no coverage but have the ability to contest other nodes due to their reach.*

(ii) *Let $B \subset N$ be a block in the network D. B must contain all nodes that either contest each other for at least one $(i, j) \in Cov_B(D)$.*

Redundancy of blocks

There can exist a large number of blocks in any network. In fact, the number increases proportionally with the number of structural holes (Burt, 2002) between nodes in the network. However, not all of the blocks in a given network are equally compelling. There can exist blocks that are *redundant*.

Blocks that are redundant contain members who when removed from the block do not reduce its brokerage abilities in the given network and therefore do not facilitate its ability to broker relationships in any way. More accurately, a block is redundant if a subset of the block is still uncontested in the brokerage of the same relations. This notion is provided more formally in Definition 2.9.

Definition 2.9 *Let D be a connected network on node set N such that $B \subset N$ is a block in D. Furthermore, let $i, j \in N$.*

(a) *The **brokerage set** of node set B is given by*

$$\mathcal{Z}_B(D) = \left\{ (i, j) \in Cov_B(D) \,\middle|\, \mathcal{W}_{ij}(D - B) = \varnothing \right\}. \tag{6}$$

(b) *Block B is **redundant** if there exists some $B' \subset B$ such that $\mathcal{Z}_{B'}(D) \supseteq \mathcal{Z}_B(D)$.*

Further properties can be shown regarding the configuration of blocks and their relatedness to the notions of coverage and contestability introduced above.

Properties 2.10 *Let D be a connected network on node set N.*

(i) *Suppose $i \in N \setminus \mathcal{M}(D)$ is fully contested by some node set $C_i \subset N$. Let $B = C_i \cup \{i\}$. Then $B \in \mathcal{B}(D)$ if and only if there exists $(j,k) \in Cov_h(D)$ for some $h \in B$ with $j,k \notin B$.*

(ii) *A block $B \in \mathcal{B}(D)$ is not redundant if and only if $\mathcal{Z}_B(D) \subset Cov_i(D - (B \setminus \{i\}))$ for every $i \in B$.*

(iii) *All blocks containing a source and/or a sink are redundant.*

(iv) *Let $\mathcal{B}(D) \neq \varnothing$. Now $\mathcal{B}_i(D) = \varnothing$ implies that $i \in \bigcap_{B \in \mathcal{B}(D)} \mathcal{Z}_B(D)$.*

(v) *Let $B, B' \in \mathcal{B}(D)$ be two distinct blocks in D. The union $B'' = B \cup B'$ is not a block if and only if $\mathcal{Z}_B(D) \subseteq B'$ and $\mathcal{Z}_{B'}(D) \subseteq B$.*

(vi) *If $i \in N$ is uncontested, then $Cov_i(D) = \mathcal{Z}_i(D)$.*

The relationship between redundancy and contestability is non-trivial: A non-redundant block can contain members that do not either fully or partially contest each other; and a non-redundant block can contain members where all of them either fully or partially contest each other.

Property 2.10 (iii) notes a relationship between coverage and redundancy, suggesting that all members of the block must have a coverage that coincides with the brokerage set of the block, even given the removal of all other members of the block. Property 2.10 (iv) naturally follows since neither sinks nor sources have a coverage in the network and therefore must be redundant to a blocks brokerage. Finally, Property 2.10 (v) is a condition that becomes important with respect to the block formation game illustrated in section **??** below.

3 A block formation game

The interdependence between the structure of a network and the formation of blocks is expressed through the use of a block formation game. Here, players maximise their individual network power by either exploiting their own position or signalling to others in an effort to form a block. The strategies of each player are informed by the each player's block set and the payoffs to each strategy is a function of the perceived per capita network power of the pursued block minus the cost of signalling to the other members of the block.

The formation of a block requires consent from all members. As such the block formation game described is considered to be an augmented version of Myerson (1991) network formation game. We argue that this is the most natural format to describe the process of block formation: If there exists no consent between players then the block becomes dysfunctional and its exploitive properties are nullified. Some characteristics from Myerson's game remain, however by implementing the game on an existing network some new characteristics are observed. Specifically we note that individual beliefs and expectations can be formed from knowledge of the networks topology which leads to more convincing equilibria.

3.1 Setting up the game

The block formation game (A, π, D) is structured as a non–cooperative, strategic form game on the node set $N = \{1, \ldots, n\}$ for a given directed network D on N. Nodes represent players, who pursue the optimisation of competitive control in the network through the participation of blocks.[2] The structure of the block formation game is given below:

- The action set for every player $i \in N$ is given by the following:

$$A_i = \mathcal{B}_i(D) \cup \{i\}, \tag{7}$$

  If $a_i = B \in \mathcal{B}_i(D)$ then player i signals to all $j \in B$, where $i \neq j$, her willingness to form B. If $a_i = i$ then agent i chooses to only exploit her own position. An action $a_i \in A_i$ is *stable* if and only if $a_j = a_i$ for all $j \in a_i$.
  A strategy tuple $a = (a_1, \ldots, a_n) \in A$ results into the block structure

$$\mathcal{A}^0(a) = \{B \in \mathcal{B}(D) \mid a_i = B \text{ for every } i \in B\}. \tag{8}$$

  We denote by $\mathcal{A}(a) = \mathcal{A}^0(a) \cup \{\{i\} \mid i \in N \setminus (\cup \mathcal{A}^0(a))\}$ the corresponding partitioning of N that results under $a \in A$.

[2]We recall that a non-cooperative game in normal form is given as a triple (N, A, π), where for every individual $i \in N$, A_i denotes her action set, such that $A = \Pi_{i \in N} A_i$, and $\pi_i \colon A \to \mathbb{R}$ denotes her payoff function.

- Let $\mathcal{A} = \bigcup_{i \in N} A_i \equiv \mathcal{B}(D) \cup N$ be the union of all action sets and let $\sigma \colon \mathcal{A} \to \mathbb{R}$ be some network power measure. Thus, $\sigma(B)$ denotes the power that some $B \in \mathcal{A}$ exerts in the flow structure represented by the directed network D.
  The payoff function for some $i \in N$ is given as:

$$\pi_i(a) = \delta_{a_i}(a) \cdot \frac{\sigma(a_i)}{\#a_i} - (\#a_i - 1)c, \tag{9}$$

  where $\sigma(a_i) \in \mathbb{R}$ is the network power of the selected block $a_i = B \in \mathcal{A}$, $c \geqslant 0$ is a cost of sending a signal to the other members in the selected block $a_i = B$, and $\delta_{a_i}(a)$ is a Kronecker indicator function for the action a defined by

$$\delta_B(a) = \begin{cases} 1 & \text{if } a_i = B \text{ for every } i \in B \\ 0 & \text{otherwise.} \end{cases} \tag{10}$$

The block formation game represents that nodes as decision makers aim to optimise the returns on the blocking activities they participate in. We assume that blocking is exclusive and nodes can only participate in a single block.

The payoff function assumes an egalitarian distribution of the network power of the block to all of its members. All nodes are assumed to pay for the coordination of their blocking activities and to pay a message cost of $c \geqslant 0$ to each other member of the selected block. Blocks only emerge when all of its constituting nodes consent and agree to participate.

It is costless to remain independent: In the case, $a_i = i$ the resulting payoff is simply given by $\sigma(\{i\})$.

Potential function

The block formation game can be written as an exact potential game in which the incentive of all players to change their strategy is expressed using a single global function—the potential function. The potential function of the block formation game is given by:

$$\Phi(a) = \sum_{S \in \mathcal{A}(a)} \frac{\sigma(S)}{\#S} - \sum_{i \in N} \#a_i \cdot c,$$

where $\mathcal{A}(a) = \{S \in \mathcal{A} \mid a_i = S \,\forall\, i \in S\}$ resulting from the action tuple $a = (a_{-i}, a_i)$. In following the payoff function for the game, the potential function aggregates the network power per capita of each block and individual position that is formed and exploited, and subtracts the cumulative costs in pursuing the actions in a across all $i \in N$.

Let $a = (a_{-i}, a_i)$ and $a' = (a_{-i}, a'_i)$ be action tuples for the game (A, π, D) where $a'_i \in A_i \setminus a_i$ is some other action for $i \in N$. The difference between action tuples is only the action of player i changing from a_i to a'_i. The change in the potential function is given as:

$$\Phi(a) - \Phi(a') = \frac{\sigma(a_i)}{\#a_i} - \frac{\sigma(a'_i)}{\#a'_i} - (\#a_i - \#a'_i)c.$$

The *maximal individual payoff* for a given action, $a_i \in A_i$, is given by:

$$\varphi(a_i) = \sigma_i(a_i) - (\#a_i - 1)c,$$

where $\sigma_i(a_i) = \frac{\sigma(a_i)}{\#a_i}$, which is the individual network power of the action a_i. The change in individual i's payoff from a change in i's acton from a_i to a'_i is given by:

$$\varphi(a) - \varphi(a') = \sigma_i(a_i) - \sigma_i(a'_i) - (\#a_i - \#a'_i)c.$$

Since the individual benefit of some action is given by the network power of the action divided equally over all members of the action the following equality holds:

$$\Phi(a) - \Phi(a') = \varphi(a) - \varphi(a').$$

The equality satisfies the condition required for an exact potential game, showing that the change in an agents utility from a change in their individual action is exactly equal to the change in the potential function when the individual action is changed.

3.2 Equilibrium analysis

We characterise the equilibrium of the game under a number of different equilibrium concepts: strong Nash equilibrium, Nash equilibrium, and monadic stability. Note that the analysis of the equilibrium are given assuming $c \geqslant 0$, however when c increases the equilibrium become trivial: with high costs to signalling no player will have any incentive to signal to any other player or block.

3.2.1 Strong Nash equilibrium

For any coalition $B \subset N$ and strategy profile $a \in A$ we denote by a_B the B-restriction of a defined by $(a_j)_{j \in B}$ and by $a_{N \setminus B}$ its compliment $(a_k)_{k \notin B}$.

Definition 3.1 *Consider a block formation game (A, π, D). The action tuple $\tilde{a} \in A$ is a **Strong Nash equilibrium** if for any $B \subseteq N$, where $B \neq \emptyset$, and every coordinated strategic deviation $b_B = (b_i)_{i \in N} \in A_B = \Pi_{i \in B} A_i$ it holds that $\pi_i\ (\tilde{a}_{N \setminus B}, b_B) \leqslant \pi_i(\tilde{a})$ for all nodes $i \in B$.*

To find the existence of a strong Nash equilibrium (SNE) the set $\mathcal{A}$ containing all actions is ranked in terms of the maximal individual payoff for each $S \in \mathcal{A}$, given by $\varphi(S) = \frac{\sigma(S)}{\#S} - (\#S - 1)c$ in the block formation model with egalitarian distribution given above.

We construct a ranking, $R(\varphi) \subseteq \mathcal{A}$, for a given $c \in \mathbb{R}$ as follows:

(1) Select $S^1 \in \arg\max \{\varphi(S) \mid S \in \mathcal{A}\}$.

(2) Let $S^1, \ldots, S^m$ be selected. Choose:

$$S^{max} \in \arg\max \left\{ \varphi(S) \,\middle|\, S \in \mathcal{A} \text{ where } S \subset N \setminus \bigcup_{k=1}^{m} S^k \right\}. \tag{11}$$

(3) Continue until:

$$\bigcup_{k=1}^{m} S^k = N,$$

where the outcome is the partition $R(\varphi) = \left(S^1, \ldots, S^K\right)$.

From this we can introduce a corresponding strategy tuple $\tilde{a} \in A$ by letting $\tilde{a}_i = S^m$ for every node $i \in S^m$, $m = 1, \ldots, K$. Clearly, $\mathcal{A}(\tilde{a}) = R(\varphi)$.

Theorem 3.2 *The strategy tuple $\tilde{a}$ introduced above is a Strong Nash equilibrium in a block formation game (A, π, D).*

The SNE attained leads to a set of actions, $\tilde{a} \in A$, derived from the partition $R(\varphi)$, such that $\nexists\, S, S' \in \tilde{a}$ where $S \cap S' \neq \emptyset$ due to the mutual exclusivity property. We note that no restrictions are made on the algorithm to construct the SNE such that the network power of each action $S \in \mathcal{A}$ can be any real number and therefore the individual maximal payoff can be any real number. Theorem 3.2 is immediately followed by the following propositions.

Properties 3.3 *Let $\tilde{a} \in A$ be some SNE of the block formation game (A, π, D) on network D, and $\mathcal{A} = \bigcup_{i \in N} A_i \equiv \mathcal{B}(D) \cup N$ where $S, S' \in \mathcal{A}$.*

(i) *$B \in \mathcal{A}(\tilde{a})$ if and only if there does not exist some $B' \in \mathcal{A}(\tilde{a})$ such that $B' \cap B \neq \emptyset$ and $\varphi(B') > \varphi(B)$ and, moreover, there is no $B'' \in \mathcal{A}$ such that $B'' \cap B' \neq \emptyset$ and $\varphi(B'') > \varphi(B')$.*

(ii) *$B \notin \mathcal{A}(\tilde{a})$ if and only if there does not exist some $B' \in \mathcal{A}$ such that $B \cap B' \neq \emptyset$ and $B' \in \mathcal{A}(\tilde{a})$.*

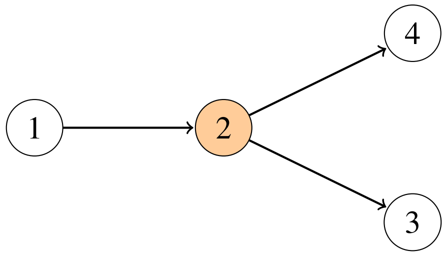

Figure 1: Directed network D with two blocks: $B = \{2,3\}$ and $B' = \{2,4\}$.

(iii) *There exists multiple SNE in the block formation game if and only if there exist two distinct $B, B' \in \mathcal{A}$ where $B \neq B'$, $\varphi(B) = \varphi(B')$, $B \cap B' \neq \emptyset$, and there is no $B'' \in \mathcal{A}$ such that $\varphi(B'') > \varphi(B)$, $B'' \cap B \neq \emptyset$, $B'' \cap B' \neq \emptyset$, and $B'' \in \mathcal{A}(\tilde{a})$.*

Multiple strong Nash equilibria can emerge if there exist multiple blocks or individual positions with equal individual payoffs, overlapping memberships, and there does not exist another block with an overlapping membership and higher individual payoffs, that exists in a strong Nash equilibrium. The example below provides a game in which there exists multiple SNE.

Example 3.4 Consider a network D on node set $N = \{1,2,3,4\}$ shown in Figure 1 such that $\mathcal{B}(D) = \{B, B'\}$, where $B = \{2,3\}$ and $B' = \{2,4\}$. Let $0 \leqslant c < 10$, $\varphi(B) = \varphi(B') = 10$, and $\varphi(1) = \varphi(2) = \varphi(3) = \varphi(4) = 0$, here there exists two SNE: (1) Where $\tilde{a_1} = (1, B, B, 4)$, and (2) Where $\tilde{a_2} = (1, B', 3, B')$. ♦

3.2.2 Nash Equilibrium

A Nash equilibrium (NE), denoted by $a^\star \in A$, satisfies the property that every individual $i \in N$ selects the *best response* to the actions selected by the other individuals. The definition of Nash equilibrium in the block formation game is given below.

Definition 3.5 *Consider a block formation game (A, π, D). The action tuple $a^\star \in A$ is a **Nash Equilibrium** if for every $i \in N$, $\pi_i(a^\star) \geqslant \pi_i(a_i, a^\star_{-i}) \, \forall \, a_i \in A_i$ where $a_i \neq a^\star_i$.*

In a block formation game there can exist multiple NE and the set of SNE are a subset of the set of NE. Theorem 3.6 notes all actions that are stable in at least one NE. Here, we note that potentially many actions are stable in NE.

Theorem 3.6 *$B \in \mathcal{A}(a^\star)$ for some Nash equilibrium $a^\star$ in the block formation game if and only if there is no $i \in B$: $\sigma(i) > \varphi(B)$.*

Given Theorem 3.6 we note that a block does not form in any NE if there exists a best response strategy for some member of the block that leads to an improvement by removing herself from the block and exploiting their own position. By elaboration, no blocks are formed in NE if: (1) $\mathcal{B}(D) = \emptyset$, or (2) $c > \frac{\sigma(B^1)}{\#B^1(\#B^1-1)}$, where $B^1 \in \arg\max \{\varphi(B) \mid B \in \mathcal{B}\}$, or (3) $\forall B \in \mathcal{B}(D)\, \exists\, i \in B$ such that $\varphi(i) > \varphi(B)$. The above also equally apply to SNE.

Some agents remain stubborn to participating in a block; these agents will determine the equilibria that emerge. Specifically, they only wish to strictly form a block with other agents if the formation of the block provides a strictly higher maximal individual payoff than the exploitation of the agents inherent position.

The NE that emerge are analogous to that of Myerson's network formation based on consent (Myerson, 1991). A note is made regarding *Myerson's Lemma.* Myerson's Lemma states that in a consensual network formation game the empty signalling profile for all agents is a NE due to the consent required of forming a link. Consequentially, the empty network is always an equilibrium. When applied to the block formation game this Lemma suggests that there always exists a NE where no blocks emerge irrespective of whether or not their formation provides all of its members a payoff larger than the payoff of exploiting each members individual position in the network. This is noted in part (b) of the following corollary.

Corollary 3.7 *Let $i \in N$ and consider $B \in A_i$.*

(a) *B is strictly dominated by $i \in A_i$ if and only if $\sigma(i) > \varphi(B)$*

(b) *B is weakly dominated by $i \in A_i$ if and only if $\sigma(i) \geqslant \varphi(S)$.*

(c) *The strategy tuple a^0 defined by $a^0_i = i$ for every $i \in N$ is always a Nash equilibrium.*

We remark that, due to the consent required for the formation of blocks, there exist multiple NE in (A, π, D) if $\mathcal{B}(D) \neq \emptyset$ where $B \in \mathcal{B}(D)$ is such that $\varphi(B) \geqslant \varphi(i)$ for all $i \in B$. This suggests that there exists an extremely weak condition for multiple Nash equilibrium to emerge in a block formation game.

Since the block formation game can be perceived as an augmented network formation game based on consent all resulting NE are equivalent to that of *link deletion proof.* In equilibrium no agent will have any incentive to remove itself from a block and therefore sever its relationships with other members of the block. However, when considering NE the network does not necessarily satisfy Pairwise Stability due to the requirement of consent for all agents within a block (Jackson and Wolinsky, 1996).

Potential maximisers. A potential maximiser of the block formation game refers to the set of strategy tuples that maximise the potential function with respect to the property of mutual exclusivity in that there cannot exist stable actions in a given equilibrium such that there is overlapping membership in these actions. We note that the potential maximiser for any block formation game will always be equivalent to a NE, not always a SNE. There may exist some instances in which the potential maximiser is equal to an SNE, but this is not the rule.

This result becomes obvious when noting that the SNE and the potential maximiser are identifying two different conditions. On one hand the potential maximiser is identifying the greatest breadth of the number of blocks in the network and the SNE condition is identifying the greatest the blocks that provide maximal attainable payoffs for all of its members.

In order for each player to know their own action set and the network power, and therefore the payoff, to each action the complete structure of the network must be known by all players. If the structure of the network is known by all players then each player can derive each others action set and incentives, and from this build an expected payoff function to each action they can participate in. From this information each agent can predict with some probability the actions others will take and can therefore coordinate their signalling strategies in a non–cooperative way to attain the highest payoff.

This form of farsighted network formation is expressed through the equilibrium concept of *monadic stability*, seminally introduced by Gilles and Sarangi (2010).

3.2.3 Monadic stability

Gilles and Sarangi develop a *belief–based* stability concept for understanding a purely non–cooperative process of limited farsighted network formation with positive costs. The stability concept is termed as monadic stability (MS). Under MS an individual assumes that other individuals are likely to respond affirmatively to a proposal to form a link if the addition of this link is profitable for them. The belief system provides a degree of realism to the network formation model: from viewing other individuals characteristics one can gain information of the other, or set of others, and use this to base their belief on whether a relationship will be formed and a signal reciprocated.

We provide a version of the MS concept in which individual agents form a belief regarding the pursued actions of other agents, and make their actions based on the expectation that a given block will be formed. By knowing the structure of the network only players can derive information which is used to inform the beliefs of what the other players actions will be, albeit in a relatively myopic fashion.

We initially introduce some new notation used to elaborate on the formation of beliefs considered below. First, we let the set $\mathcal{H}_i(S)$ denote the set of player i's *superior* actions to some action $S \in A_i$, which is given by

$$\mathcal{H}_i(S) = \{S' \in A_i \mid \varphi(S') > \varphi(S)\} .$$

Second, we let the set $O_i(S)$ denote the set of player i's *rival* actions to some action $S \in A_i$, which is given by

$$O_i(S) = \{S'' \in A_i \mid \varphi(S'') = \varphi(S)\} .$$

In Monadic Stability it is assumed that players are myopically rational in that they derive a probabilistic belief that some action will be stable which subsequently informs their expected payoff and equilibrium action. Let Γ be the profile of beliefs for all players in which $\Gamma_i(S)$ is the belief that player i has in the stability of action $S \in \mathcal{A}$ in equilibrium and Γ_i be a profile of i's beliefs regarding all $S \in \mathcal{A}$. Each player, $i \in N$, pursues an action that maximises their expected payoff based on Γ_i. The expected payoff of player i for some $S \in A_i$ is given by

$$\mathbb{E}\left[\pi_i(S)\right] = \Gamma_i(S) \cdot \left(\frac{\sigma(S)}{\#S}\right) - (\#S - 1)c \equiv \Gamma_i(S) \cdot \varphi(S),$$

where $\Gamma_i(S)$ is player i's probabilistic belief that action a_i will be stable in equilibrium. The expected payoff function weights the individual network power derived from a given action by the probability that all other agents required to pursue the action will also pursue the action based on their beliefs. The costs to pursuing the action are imposed regardless of whether or not an action is stable.

Formally, we derive player i's belief that a given action, $S \in \mathcal{A}$, will be stable in equilibrium as

$$\Gamma_i(S) = \prod_{j \in S : i \neq j} \left[\frac{1 - \sum_{S' \in \mathcal{H}_j(S)} \Gamma_j(S')}{1 + \sum_{S'' \in O_j(S)} \gamma_j(S'')}\right],$$

where

$$\gamma_j(S'') = \prod_{h \in S''} \left[1 - \sum_{S^\circ \in \mathcal{H}_h(S'')} \Gamma_h(S^\circ)\right].$$

Therefore, the belief of player i that action S will be stable depends the probabilistic actions of all $j \in S$ where $i \neq j$. Given the known structure of the network, and thus the public nature of the payoffs to actions, the probability that all superior and rival actions of j can be calculated which in turn informs i's assessment of whether S will be stable. Note that if $S = i$ for some $i \in N$ then $\mathbb{E}\left[\pi_i(i)\right] = \pi_i(i) = \sigma(i)$ since the result of an empty product is 1.

Determining myopic beliefs. Players calculate their myopic beliefs in a mechanistic fashion similar to the SNE algorithm above, beginning with the assessment of the action that provides the highest maximal individual payoff then continuing. Below we provide the procedure for calculating player i's myopic beliefs regarding the stability of some $S \in \mathcal{A}$.

(1) Let $F = \mathcal{S}^1 \cup \ldots \cup \mathcal{S}^{m-1}$ be assessed already.

(2) Select $S^m \in \arg\max \{\varphi(S) \mid S \in \mathcal{A} \setminus F\}$. There may exist a set of actions that have the same maximal individual payoff given by $\mathcal{S}^m = \{S \in \mathcal{A} \setminus F \mid \varphi(S) = \varphi(S^m)\}$.

(3) Calculate

$$\Gamma_i(S^m) = \prod_{j \in S^m : i \neq j} \left[\frac{1 - \sum_{S' \in \mathcal{H}_j(S^m)} \Gamma_j(S')}{1 + \sum_{S'' \in O_j(S^m)} \gamma_j(S'')} \right]$$

for all $S^m \in \mathcal{S}^m$.

(4) Continue until $\mathcal{A} \setminus F = \varnothing$.

(5) Each player chooses an action $S \in A_i$ that maximises : $\mathbb{E}[\pi_i(S)] = \Gamma_i(S) \cdot \varphi(S)$. If there exists a set $\mathcal{S}_i^\star \subset A_i$ of actions that maximise i's expected utility based on her myopic beliefs then i pursues some $S \in \mathcal{S}_i^\star$ with a given expectation of $\frac{1}{\#\mathcal{S}_i^\star}$.

There exists multiple actions for any player that maximise the players expected payoff given other players beliefs if and only if there exists some $\Gamma_i(S)$ where $0 < \Gamma_i(S) < 1$ for any $i \in N$.

We extend the analysis by distinguishing between so–called *pure* beliefs and *mixed* beliefs below.

Definition 3.8 *Let (A, π, D) be a block formation game on network D and player set N.*

(a) *Player $i \in N$ has a **pure belief** that some action $S \in \mathcal{A}$ will be stable if and only if $\Gamma_i(S) \in \{0, 1\}$. Player $i \in N$ has a **mixed belief** that some action $S \in \mathcal{A}$ will be stable if and only if $0 < \Gamma_i(S) < 1$.*

(b) *A game is a **pure game** if and only if, for all $i \in N$, $\Gamma_i(S) \in \{0, 1\}$ for all $S \in \mathcal{A}$.*

An agent with a pure belief has a single strategy that maximises their expected payoffs and will therefore choose this strategy .

Definition 3.9 *Consider a block formation game (A, π, D) on network D and player set N. An action profile $\hat{a} \in A$ is* ***monadically stable*** *if all $i \in N$ select $\hat{a}_i \in A_i$ that maximises their expected payoff with respect to Γ_i, and their myopic beliefs are confirmed in that all actions pursued by all i are stable in $\hat{a}$.*

A number of theorems follow from the definition of MS above which characterise the equilibrium.

Theorem 3.10 *Let (A, π, D) be a block formation game on network D and node set N. If (A, π, D) is a pure game then the resulting equilibrium will be both MS and SNE.*

The proof of Theorem 3.10 is relatively easy to understand. If all $\Gamma_i(S) = 1$ for all $i \in N$ and $S \in \mathcal{A}$ the process for forming myopic beliefs equates exactly to the algorithm for calculating the SNE of the block formation game. We note that there can exist MS equilibrium that do not rely on the necessity for pure beliefs for all players. However these will depend on some mixed strategy NE.

4 Application: Brokerage power

Section 2 provided the notions of middlemen, blocks, and contestability, which, as of yet, have had a limited role in the analysis of block formation. We suggest that middlemen and blocks attained power through their ability to broker relations that are only able to be negotiated by these node sets. Middlemen and blocks therefore monopolise indirect relationships between players that would not be served if the middlemen or blocks were removed from the network. Below we introduce a network power measure based on brokerage and from this we structure the power of an action based on the brokerage index of the action. Through the analysis of the game we find that players attempt to form blocks with others in an effort to increase their collective brokerage, and thus their individual payoff.

4.1 A brokerage index

All middlemen and blocks have a degree of power that is derived from their unique position, coverage, and thus their ability to broker interaction between pairs of nodes in the network. A general measure of the brokerage power of a node set is given below.

Definition 4.1 *Let D be a network on node set $N = \{1, \ldots, n\}$ and let $B \subset N$. The* ***brokerage index*** *of node set B is defined as*

$$\tau_B(D) = \#\mathcal{Z}_B(D) \equiv \sum_{i \in N \setminus B} \# [S_i(D) \setminus B] - \sum_{i \in N} \#S_i(D - B). \tag{12}$$

We refrain from normalising the brokerage index in this article. A normalisation of the index is provided in Sims and Gilles (2014).

Proposition 4.2 *Let D be a network on node set $N = \{1, \dots, n\}$ where $B \subset N$ is some node set.*

(i) *For all B it holds that $0 \leqslant \tau_B(D) \leqslant (n-1)(n-2)$.*

(ii) *$\tau_B(D) > 0$ if and only if $B \in \mathcal{B}^\star(D)$.*

Following from the proposition, if $\tau_B(D) = 0$ then the node set $B \subset N$ does not broker any relationships and must not be either a block or a middleman. The maximum brokerage, $\tau_B(D) = (n-1)(n-2)$, is given to a node set consisting of a single node at the centre of an undirected star. In a directed cycle of size n, $\tau_i(D) = \frac{(n-1)(n-2)}{2}$ for all $i \in N$; in an undirected chain of length n, $\tau_i(D) = 2\,[\#P_i(D) \cdot \#S_i(D)]$ for all $i \in N$; and in a directed chain of length n, $\tau_i(D) = \#P_i(D) \cdot \#S_i(D)$ for all nodes $i \in N$.

The brokerage index can be modified and used in different networks and for different applications. We provide two modifications of the brokerage index in Appendix A and suggest applications for these modified metrics. The first modified metric integrates closeness centrality with the brokerage index and the second measure provides some application of the brokerage index to weighted networks.

Application to block formation

We denote the block formation game based on brokerage in the network D and node set N as $\left(A^b, \pi^b, D\right)$. The power of a node set is now defined in terms of its brokerage index (Equation 12), implying that the brokerage of node set $B \subset N$ is $\sigma(B) = \tau_B(D)$. In the general analysis above there existed nothing to distinguish middlemen from other players: we allowed $\sigma(S)$, where $S \in \mathcal{A} = \mathcal{B}(D) \cup N$, to take any real number. From the brokerage measure there exists a quantitative difference between middlemen and other players. As above we note that the *maximal individual payoff* for some action $a_i \in A_i^b$ is for all $i \in a_i$ given by

$$\varphi(a_i) = \frac{\tau_{a_i}(D)}{\#a_i} - (\#a_i - 1)c \tag{13}$$

Considering the brokerage function, the maximal individual payoff for each action becomes restricted, such that:

- $\varphi(j) = 0 \,\forall\, j \notin \mathcal{M}(D)$;
- $\varphi(i) \geqslant 1 \,\forall\, i \in \mathcal{M}(D)$; and

- $\varphi(B) = \frac{\sigma_B(D)}{\#B} - (\#B - 1)c$ for $B \in \mathcal{B}(D)$.

Middlemen are always able to secure some positive payoff irrespective of the actions of other agents and non–middlemen will choose to exploit their position if and only if they cannot be a member of any block, $B \in \mathcal{B}(D)$, where $\varphi(B) \geqslant 0$. Already, given the equilibrium analysis in Section **??**, we can stipulate that the incentives and actions of middlemen dominate the SNE and NE that emerge in any block formation game based on brokerage.

4.2 Equilibrium analysis

Strong Nash equilibrium

To find the set of SNE in the block formation game based on brokerage the set of all actions are ranked in terms of their respective maximal individual payoff to produce a ranked partition. The partition then corresponds exactly to the actions of all payers in the game as noted in Theorem 3.2. However, due to the aforementioned restrictions on the maximal individual payoffs of blocks and individual positions provided by the brokerage measure a more efficient algorithm for calculating the SNE can be derived.

Let $\mathcal{B}^\circ = \{B \mid B \in \mathcal{B}^\star(D) \text{ with } \varphi(B) > 0\}$. We construct a partition, $R(\varphi) \subseteq \mathcal{B}^\circ$, for a given $c \geqslant 0$ as follows:

(1) Select $B^1 \in \arg\max \{\varphi(B) \mid B \in \mathcal{B}^\circ\}$.

(2) Let $B^1, \ldots, B^m$ be selected. Choose:

$$B^{max} \in \arg\max \left\{ \varphi(B) \middle| B \in \mathcal{B}^\circ, B \subseteq N \setminus \bigcup_{k=1}^{m} B^k \right\}. \tag{14}$$

(3) Continue until:

$$\arg\max \left\{ \varphi(B) \middle| B \in \mathcal{B}^\circ, B \subseteq N \setminus \bigcup_{k=1}^{m} B^k \right\} = \varnothing. \tag{15}$$

Where the outcome is $R(\varphi) = \left(B^1, \ldots, B^K\right)$.

From this we define $\tilde{a} \in A^b$ for $\left(B^1, \ldots, B^K\right)$ by: $\tilde{a}_i = B^m \,\forall\, i \in B^m$ and $\tilde{a}_j = j \,\forall\, j \in N \setminus \bigcup_{k=1}^{K}$. This provides the SNE of the block formation game based on brokerage and provides a more efficient mechanism to reach a SNE than the more general algorithm for calculating SNE above.

Theorem 4.3 *Let $\tilde{a} \in A^b$ be a SNE of a block formation game $\left(A^b, \pi^b, D\right)$ based on the brokerage index τ. All blocks in $\tilde{a}$ are non–redundant.*

Non–middlemen always have incentives to form blocks with other nodes provided that the cost for forming the block does not outweigh the individual payoff. However, it is notable that under some circumstances middlemen have an incentive to form blocks with other players if and only if the players partially contest each other in some way: these other players could equally be middlemen or non–middlemen. The condition to which nodes wish to form blocks with other nodes is expressed in Theorem 4.5 (a) below. Before elaborating on the theorem we first analyse the NE that emerges from the block formation game applied to brokerage.

Nash equilibrium

At least as many blocks emerge in the NE as in the SNE. Specifically the number of NE is proportional to the number of blocks in the networks block set. With respect to brokerage we note that any block has the potential to emerge in NE if and only if the block does not contain a middleman that can earn a maximal individual payoff from exploiting her own position that is at least equal to or higher than the payoff from participating in the block. This is a characteristic that is shared with SNE and is explicitly shown with respect to Example 4.7 below: as the cost of sending individual signals rises from 0 to 1, middlemen 2 and 5 have incentives to deviate from their blocks B and B' respectively. This outcome is directly derived from Theorem 3.6. Subsequently we note the following corollary.

Corollary 4.4 *Let $\left(A^b, \pi^b, D\right)$ be a block formation game based on the brokerage index τ and let the action set $a^\star \in A^b$ be some NE.*

(a) *Block $B \notin a^\star$ if and only if there exists some $i \in B$ such that $\varphi(i) > \varphi(B)$.*

(b) *Both redundant and non–redundant blocks are Nash stable.*

It is also notable in Example 4.7 that player 6 has no incentive to form a block even though $\mathcal{B}_6(D) \neq \varnothing$. Player 6 is uncontested, i.e., neither partially contested or fully contested, and will therefore never have a strict preference to form a block with any other player. The decision for some player to form a block with others is thoroughly explained in Theorem 4.5 below.

Theorem 4.5 *Let $\left(A^b, \pi^b, D\right)$ denote the block formation game based on the brokerage index τ on the network D. Consider a set of relationships, $K_i \subseteq Cov_i(D) \setminus \mathcal{Z}_i(D)$, for some $i \in N$ where $K_i \neq \varnothing$.*

Player i must have a corresponding node set, $C_i \subset N \setminus \{i\}$, where $K_i \subseteq \bigcup_{j \in C_i}(\overline{P}_j(D - i) \times S_j(D - i)$ and $C_i \cup \{i\} = B \in \mathcal{B}(D)$, where $K_i \subseteq \mathcal{Z}_B(D)$.

If $\nexists\, B \in \mathcal{B}_i$ where $(\#B - 1)(\#Bc + \#\mathcal{Z}_i(D)) \leqslant \#K_i \,\forall\, K_i \subseteq Cov_i(D) \setminus \mathcal{Z}_i(D)$, then i will always wish to exploit her position only.

The theorem highlights when players wish to form blocks with others. If the condition is satisfied in part (a) of the theorem then player i's dominant strategy is to exploit her middleman position and any block that the middleman is a member of will not form in any NE. Part (b) of the theorem implies that if i is uncontested and $c > 0$ then there is no $B \in \mathcal{B}_i(D)$ that is Nash stable and therefore Strongly Nash stable. Indeed, if $c > 0$ then it must be that $\varphi(i) > \varphi(B)$ for all $i \in B$ since there exists a positive cost of signalling to form a block and, if there is some $i \in B$ that is uncontested, the absolute maximum value for $\varphi(B) = \varphi(i)$ if and only if $c = 0$. Furthermore, part (b) highlights a special case of part (a): since i is uncontested it must be that $K_i = \text{Cov}_i(D) \setminus \mathcal{Z}_i(D) = \varnothing$, therefore there does not exist any $C_i \subset N \setminus \{i\}$ that can fully contest i with respect to K_i because K_i does not exist. In this case i will have no strict incentive to form a block with another set of players to improve her individual brokerage, only a weak incentive if and only if all i that comprise the block have the same brokerage index and $c = 0$.

Middlemen provide a key role in identifying the blocks that are stable in a NE. We have identified that middlemen who are uncontested will never have an incentive to participate in a block; only middleman who are partially contested will have an incentive to form a block if and only if the maximal individual payoff of forming the block exceeds the maximal individual payoff of exploiting her middleman position.

We conclude by noting that both redundant and non–redundant blocks have an ability to form due to the consent required to form a block and that the SNE are a subset of the NE of a given block formation game. Arguably the SNE concept leads to a more realistic equilibrium than NE. Below we add more realism to the equilibrium by applying the concept of monadic stability to brokerage.

A note on uncontested nodes. A node is uncontested if and only if it is not either fully or partially contested by any other set of nodes. This leads to the properties that an uncontested node must be a middleman and that the uncontested node does not either partially or fully contest any other node, or set of nodes. Indeed, an uncontested node is one that negotiates relationships and has a position in the network that is truly unique. Due to these properties it is very difficult for an uncontested node to participate in a block under NE, and it is even rarer in SNE. We develop a theorem based on the properties of uncontested nodes that are applied to both SNE and NE.

Theorem 4.6 *Consider some network D on node set N where $B \in \mathcal{B}(D)$ and $\exists\, i \in B$ such that i is and uncontested.*

(a) *If $\varphi(i) < \varphi(B)$ then B is not stable in any SNE.*

(b) *If $\varphi(i) > \varphi(B)$ then B is not stable in any NE or SNE.*

Following from Theorem 4.6 it is therefore only plausible that an uncontested node will only participate in a block if $\varphi(i) = \varphi(B)$. Obviously, however, the uncontested node will be indifferent between participating in the block and operating individually.

An illustrative example

We provide an example that illustrates the equilibrium concepts with respect to the block formation model based on the brokerage index τ.

Example 4.7 Consider network D on node set $N = \{1, 2, 3, 4, 5, 6, 7\}$ shown in Figure 2. Players 2, 5, and 6 are middlemen and there exists 34 distinct blocks, however only 3 of these blocks are non–redundant.

- **For $\mathbf{0 \leqslant c < 1}$**

  There exists a unique SNE which refers to the formation of blocks $B = \{2, 3\}$ and $B' = \{4, 5\}$ and $a_i = i$ for all $i \in \{1, 6, 7\} \equiv N \setminus \{2, 3, 4, 5\}$. Player 6 will never have any incentive to form a block since it is uncontested.
  There exist multiple NE. Without going through all different combinations of NE we can instead note that the only blocks that can form in a NE are $B = \{2, 3\}$, $B' = \{4, 5\}$, $B'' = \{2, 5\}$ is stable if $c \leqslant 0.5$, and $B''' = \{2, 3, 4\}$ is stable if $c = 0$. Block B'' is notable as it consists of middlemen only, and block B''' is notable as it is non–redundant and still stable in NE.
  Under MS we note that the beliefs for each member are all pure and given by: $\omega_1(1) = \omega_2(B) = \omega_3(B) = \omega_4(B') = \omega_5(B') = \omega_6(6) = \omega_7(7) = 1$, where $B = \{2, 3\}$ and $B' = \{4, 5\}$. The MS equilibrium corresponds exactly to the SNE when $0 \leqslant c < 1$.

- **For $\mathbf{c = 1}$**

  There exist four SNE: (1) As above, where $B = \{2, 3\}$ and $B' = \{4, 5\}$, are stable and $a_i = i \,\forall\, i \in N \setminus \{2, 3, 4, 5\}$; (2) Where $B = \{2, 3\}$ is stable and $a_i = i \,\forall\, i \in N \setminus \{2, 3\}$ is a SNE; (3) Where $B = \{4, 5\}$ is stable and $a_i = i \,\forall\, i \in N \setminus \{4, 5\}$ is a SNE; and (4) Where $\tilde{a}_i = i \,\forall\, i \in N$.

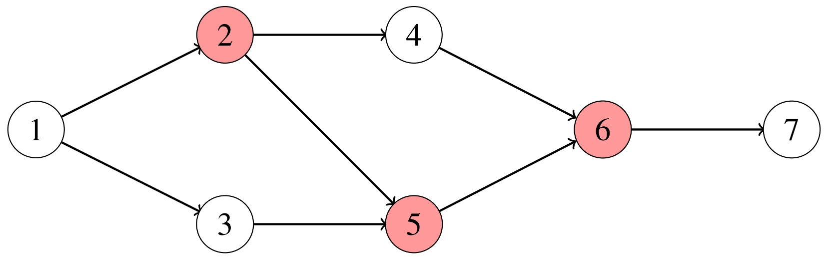

Figure 2: Acyclic directed network, D, where $\mathcal{M}(D) = \{2, 5, 6\}$ coloured in red.

In this case the set of NE are equal to the SNE. Indeed, only two blocks are stable in NE: $B = \{2, 3\}$ and $B' = \{4, 5\}$. Obviously the situation in which all agents exploit their own network position is also a NE.

Here we note that under MS agents still have pure beliefs, which are given by: $\omega_1(1) = \omega_2(2) = \omega_3(3) = \omega_4(4) = \omega_5(5) = \omega_6(6) = \omega_7(7) = 1$. Specifically, since no agent strictly benefits from forming a block they will never risk signalling to any other player in the network. Instead the expected payoff is maximised when exploiting ones own position only.

- **For c > 1**

  There exists a unique SNE where $a_i = i$ for every $i \in N$. If $c > 1$ then players 2 and 5 strictly prefer to exploit their own middleman positions as opposed to participating in some block. Under this situation the only agents that earn a payoff above zero are the middlemen. Again, the NE is equal to the SNE such that all agents exploit their own position only.

  Under MS the beliefs for each member are all pure and given by: $\omega_1(1) = \omega_2(2) = \omega_3(3) = \omega_4(4) = \omega_5(5) = \omega_6(6) = \omega_7(7) = 1$. Again, the the MS equilibrium corresponds exactly to an SNE when $c \geqslant 1$: all players exploit their own position only.

The example highlights a number of points made through the discussion. First, that non–redundant blocks can emerge in NE but not SNE. Second, that that a block formation game where all players have pure beliefs leads to an equilibrium that is also a SNE. And third, that blocks can contain solely middlemen, or non–middlemen, or a combination of both. ♦

4.3 A comparison with centrality measures

We allow the power of actions, $\sigma(a_i)$ where $a_i \in A_i$ for some $i \in N$, to be determined by other known centrality measures. In such a case we find no common measures of centrality that strictly leads to the formation of blocks or even coalitions of players. Such a finding can be logically explained. Centrality measures that indicate power must favour larger node sets in some way: the degree, betweenness, closeness, Katz–Bonacich, and the β–measure (Brink and Gilles, 1994, 2000) centralities are all measures that do not give a proportionally larger weight to larger node sets. Without some way in which to provide more power to node sets the formation of blocks will never be strictly rational. The formation of a block may however be weakly rational if all players in the block have the same centrality that is independent of each other and the cost of forming the block is zero. For example, consider two or more players that have an equal degree where none of the neighbours of the players overlap and the cost of forming a block, or a coalition of these players, is zero; in such a case the formation of a block is weakly rational only.

Rough measures for brokerage can be created that take into consideration the predecessor set and successor set of individual players and coalitions. We provide two examples below which derive from some network D on node set N where $B \subseteq N$. First, let the power measure of a block be the product of the number of direct predecessors of the node set and the number of its direct successors

$$\sigma'(B) = \left[\# \bigcup_{j \in B} p_j(D) \right] \cdot \left[\# \bigcup_{j \in B} s_j(D) \right], \tag{16}$$

where $\bigcup_{j \in B} p_j(D)$ refers to all the direct predecessors of node set B, and $\bigcup_{j \in B} s_j(D)$ refers to all the direct successors. Essentially, this is calculating the product of the in–degree and out–degree of the node set. Furthermore, we can extend this to include entire predecessor sets and successor sets of the node set $B \subseteq N$. Let

$$\sigma''(B) = \left[\# \bigcup_{j \in B} P_j(D) \right] \cdot \left[\# \bigcup_{j \in B} S_j(D) \right], \tag{17}$$

where $\bigcup_{j \in B} P_j(D)$ is the predecessor set for node set $B \subset N$ and let $\bigcup_{j \in B} S_j(D)$ be the relevant successor set. In such a case blocks have the ability to form due to the multiplication in the power function.

In both instances blocks can form. Consider the example below.

Example 4.8 Let D' be a network on node set $N = \{1, 2, 3, 4, 5, 6, 7\}$, as seen in figure 3. A block formation game is played where the power of some node set, $B \subseteq N$, is determined initially by $\tau(B)$ (Equation 3), then $\sigma'(B)$ (Equation 16), and then $\sigma''(B)$ (Equation 17).

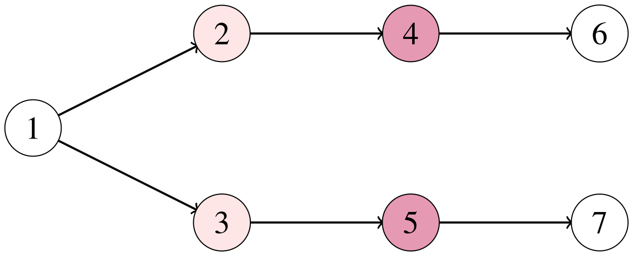

Figure 3: Acyclic directed network D'.

We consider two cases; one where $c = 0$ and one where $c > 1$. We limit our analysis to the SNE only. There exists 9 distinct blocks in D' and $\mathcal{M}(D') = \{2, 3, 4, 5\}$. We let $B = \{4, 5\}$, $B' = \{2, 3\}$, $B'' = \{2, 5\}$, and $B''' = \{3, 4\}$, and refer to these blocks throughout the example.

- **For c = 0**

  Given the initial measure, $\tau(B)$, there exists no unique SNE, indeed since $\tau(2) = \tau(3) = \tau(4) = \tau(5) = \tau(B) = \tau(B') = \tau(B'') = \tau(B''') = 2$ there can exist a SNE such that no blocks are formed, or a combination of blocks are formed that do not have any overlapping membership.

  Given $\sigma'(B)$ there can also exist multiple equilibria, however the equilibrium in which no blocks are formed is not Strong Nash here. Specifically, $\sigma'(2) = \sigma'(3) = \sigma'(4) = \sigma'(5) = \sigma'(B') = 1$ and $\sigma(B) = \sigma(B'') = \sigma(B''') = 2$. Therefore there exist only two SNE: one in which only block B is formed and all other players exploit their own position only, and one in which blocks B'' and B''' are formed and all other nodes exploit their own position only.

  Given $\sigma''(B)$ the resulting analysis of this is similar to the analysis of τ in that $\sigma''(2) = \sigma''(3) = \sigma''(4) = \sigma''(B') = \sigma''(B'') = \sigma''(B''') = 2$, however $\sigma''(B) = 3$ meaning that block B will always be in a SNE. Specifically, there exists an equilibrium in which only B is formed and all other nodes in the network exploit their own position only. Another equilibrium exist where blocks B and B' are formed and all other players exploit their own positions only.

  We can see that different payoff measures have different sets of potential SNE attached.

- **For c > 1**

  In this instance all three have the same equilibrium in that no blocks emerge and all agents exploit their own position only.

♦

Two conclusions are drawn from this example. First, the formation of blocks with σ' and σ'' is due to the multiplication nature of both of the power measures which favours larger node sets with greater coverages: if the sum of both the (direct) predecessor set and the (direct) successor set were calculated no blocks or any node sets would form. Indeed, the result would be analogous to the degree and β-measure centralities above.

Second, blocks do not form if there exists too much overlap between the predecessor sets and/or successor sets of all nodes in the block; this dilutes the incentive for node sets to form blocks. This is not the case with respect to the brokerage measure which contends that an overlap between nodes predecessor and successor sets is required for blocks to be formed. Indeed, when nodes have overlapping predecessor and successor sets it is an indication—although may not always be the case—that the nodes at least partially contest each other with respect to negotiating some relationships.

Brokerage can be combined with other centralities and mechanisms in order to get the block formation phenomenon described above. Two augmentations of the brokerage centrality measure are described in Appendix A: distance–based brokerage and brokerage on weighted networks. Although we restrict ourselves to analysis of unweighted networks with the brokerage measure given in Equation 12, the same insights hold with respect to the measures in the Appendix.

4.4 Mass and control

Using the results of brokerage and the equilibrium concepts above we provide measurements of the network that illustrate the collective power and control of the population of players that are embedded in a given topology. First, we define the mass of a network as a subset of players that are required for the negotiation of indirect relationships, elaborate on its hierarchical nature, and project this hierarchical mass into a node centrality measure. Second, we derive a control co–efficient which measures the maximal exploitation in the network.

The mass of nodes and networks

We define the blocks and middlemen that emerge under a SNE as the *mass* of the network.

Definition 4.9 *The **mass** of the network, denoted by $\mathbb{M} \subseteq N$, refers to the set of all nodes that are middlemen or members of stable blocks in SNE.*

Let $D_{\mathbb{M}} = D - \{D \cap \mathbb{M}\}$ be a restriction on the network D which only includes nodes in the mass of D, given by $\mathbb{M} \subseteq N$. Nodes within a given networks mass are important due to their power in which to broker relationships. In some cases there can exist a *hierarchy of mass* in the network such that $\mathbb{M}_1$ is the mass of the initial network D; $\mathbb{M}_2$ is the mass of the restricted network $D_{\mathbb{M}_1}$; $\mathbb{M}_3$ is the mass of the restricted network $D_{\mathbb{M}_2}$; and so on.

The hierarchy of mass is a way in which to rank both the importance and robustness of blocks and middlemen in the network. If $i \in \mathbb{M}_1$ then i has an exploitive position in the network D; if $i \in \mathbb{M}_2$ then i must also exploit those who are already exploitive in network D and therefore exploits the exploiters; if $i \in \mathbb{M}_3$ then i exploits the exploiters who already exploit the exploiters, and so on. Players in a higher mass are therefore considered to be more robust, meaning that layers of the network can be stripped away and they still maintain an exploitive position, and they are also considered to be more powerful since they directly exploit players who are already exploitive.

A node centrality measure can be developed as nodes can be ranked with respect to the highest mass that they occupy in the network. We denote μ_i as the mass of some node $i \in N$; μ_i is equal to the highest mass that i is a member of. $\mu_i = 0 \iff i \notin \mathbb{M}_1$ and $\mu_i = x$ if $i \in \mathbb{M}_x$ and $\mathbb{M}_x$ is the highest mass that i is a member of.

Example 4.10 Consider a block formation game with brokerage and where $c = 0$ on the network shown in Figure 2. The unique SNE when $c = 0$ was shown in example 4.7 to be the formation of blocks $B = \{2, 3\}$ and $B' = \{4, 5\}$, and player 6 exploits her middleman position.

The network consists of the set of players $N = \{1, 2, 3, 4, 5, 6, 7\}$ and the mass of the network D is given by $\mathbb{M}_1 = \{2, 3, 4, 5, 6\}$. In the restricted network $D_{\mathbb{M}_1}$ no blocks are formed in the block formation game with brokerage and all players exploit their own positions only. Since $\mathbb{M}(D_{\mathbb{M}_1}) = \{5\}$ the mass of the restricted network $D_{\mathbb{M}_1}$ is given by $\mathbb{M}_2 = \{5\}$. The mass of each node is given by: $\mu_1 = \mu_7 = 0$; $\mu_2 = \mu_3 = \mu_4 = \mu_6 = 1$; and $\mu_5 = 2$. ♦

If $D = \mathbb{M}_1$ or $\mathbb{M}_t = \mathbb{M}_{t+1}$ then the hierarchical process stops and all $j \in \mathbb{M}_t = \mathbb{M}_{t+1}$ are given the centrality of $\mu_j = t + 1$. An example of this occurs in any network with a directed cycle because all a nodes in the cycle exploit each other.

The mass of a network is similar to that of a *k–core*. However, whereas the k–core concept uses the degree of each node in successively restricted networks, the mass concept uses the power of node sets in successively restricted networks and forms a centrality measure based on this.

Control co–efficient

We provide a measure regarding the potential contestation of the network, i.e., a measure of the proportion of relationships that are exploited by middlemen and/or blocks, for a network topology and a given cost, $c \geqslant 0$.

Let $\tilde{a} \in A$ correspond to a SNE in a given block formation game based on brokerage (A, π, D) and with a given cost, c. Each SNE has a corresponding total payoff, given by $\pi(\tilde{a}) = \sum_{i \in N} \pi_i(\tilde{a})$. We can note the maximum total payoff by comparing the payoff over all $\tilde{a}$ for a given game:

$$\pi^{max} \in \arg\max \left\{ \pi(\tilde{a}) \,\middle|\, \pi(\tilde{a}) = \sum_{i \in N} \pi_i(\tilde{a}) \,\forall\, \tilde{a} \in A \right\}. \tag{18}$$

The *control co–efficient* for a given network, D, is given as:

$$\nu(D) = \frac{\pi^{max}}{\frac{n}{2}(n-1)(n-2)}, \tag{19}$$

where π^{max} is the maximum total payoff for the block formation game on the network D, and $\nu(D) \in [0, 1]$. As $\nu(D)$ is closer to 1 there exists more opportunities for blocks to form and middlemen to exploit their position.

The control co–efficient is a *pessimistic* perspective of the level of control on the network due to the use of the SNE which provides the maximal payoff to all agents. Alternative measures could be created that use the SNE that provides the minimum societal payoff for all nodes, or a payoff that averages across all potential SNE that can emerge.

Whereas the mass of the network provides a node centrality measure based on the power of nodes in restricted topologies, the control co–efficient provides a general overview of how exploitive a network can be if players were allowed to organise themselves into blocks and exploit middlemen positions.

5 Concluding remarks

We have provided formal definitions of blocks in networks as notions that are analogous to node cut sets. By using a measurement of power described by equation 12 as a payoff function, we provide a game in which nodes signal to others in an effort to form blocks. Blocks if and only if are formed when there is mutual consent across all of the blocks members. The Strong Nash equilibrium, Nash equilibrium, and Monadic stability equilibrium are all characterised. Under all concepts we find that middlemen play an important role in the resulting equilibrium, due to their ability to extract rents without participating in a block.

From the analysis of blocks and middlemen we measure the so–called mass of each player and the control co–efficient of the network as a whole. Whereas the mass of each player measures the importance of the node in terms of both its brokerage and its robustness, the control co–efficient provides a measure for the potential for extractive processes on the network.

Further research. There are multiple extensions and generalisations possible with regards to the model of block formation. We propose three immediate extensions. First, we note that the paper could be extended such that players can transfer utility to others in an effort to form a block. Indeed, we find that under certain circumstances players can be part of multiple blocks; utilities could be transferred between players such that one block dominates the others. Second, the current payoff function assumes an egalitarian distribution of brokerage power between all members of the block. Future work could devise a more general payoff function. Third, the structure of the network remains static across time, it may be interesting to see how the structure of the network changes as players form positions to both participate in blocks and middleman positions, and also attempt to stop themselves from being exploited. Fourth, we have noted that the formation blocks is analogous to the formation of cartels in exchange systems. However, the analysis could easily be used to see the network dynamics of related phenomena such as partnerships, mergers, and acquisitions. Indeed, we believe that we have just began the investigation of a breadth of new network–economic concepts including contestability, brokerage, and block formation.

References

BLUME, L., D. EASLEY, J. KLEINBERG, AND E. TARDOS (2009): "Trading Networks with Price Setting Agents," *Games and Economic Behavior*, 67, 36 – 50.

BRINK, R. V. D., AND R. P. GILLES (1994): "A Social Power Index for Hierarchically Structured Popuations of Economic Agents," in *Imperfections and Behavior in Economic Organizations*, ed. by R. P. Gilles, and P. H. M. Ruys, chap. 12, pp. 279–318. Kluwer Academic Publishers, Boston, MA.

BRINK, R. V. D., AND R. P. GILLES (2000): "Measuring Domination in Directed Networks," *Social Networks*, 22, 141–157.

Burt, R. S. (1992): *Structural Holes: The Social Structure of Competition.* Harvard University Press, Cambridge, MA.

——— (2002): "Bridge Decay," *Social Networks*, 24, 333–363.

Cook, K. S., R. M. Emerson, M. R. Gillmore, and T. Yamagishi (1983): "The Distribution of Power in Exchange Networks: Theory and Experimental Results," *The American Journal of Sociology*, 89(2), 275–305.

Easley, D., and J. Kleinberg (2010): *Networks, Crowds, and Markets: Reasoning about a Highly Connected World.* Cambridge University Press.

Emerson, R. M. (1962): "Power-Dependence Relations," *American Sociological Review*, 27, 31–41.

Gai, P., and S. Kapadia (2010): "Contagion in Financial Networks," *Proceedings of the Royal Society A: Mathematical, Physical, and Engineering Sciences*, 466, 2401 – 2423.

Gilles, R. P., S. Chakrabarti, S. Sarangi, and N. Badasyan (2006): "Critical Agents in Networks," *Mathematical Social Sciences*, 52, 302–310.

Gilles, R. P., and D. Diamantaras (2013): "Platform Competition as Network Contestability," *ArXiv e-prints*, (arXiv:1310.3521).

Golub, B., and M. O. Jackson (2010): "Naive Learning in Social Network and the Wisdom of Crowds," *American Economic Journal: Microeconomics*, 2(1), 112–149.

Gould, R. V. (1989): "Power and Social Structure in Community Elites," *Social Forces*, 68(2), 531–552.

Gould, R. V., and R. M. Fernandez (1989): "Structures of mediation: A formal approach to brokerage in transaction networks," *Sociological methodology*, pp. 89–126.

Goyal, S., and F. Vega-Redondo (2007): "Structural Holes in Social Networks," *Journal of Economic Theory*, 137, 460–492.

Granovetter, M. (1973): "The Strength of Weak Ties," *American Journal of Sociology*, 78, 1360–1380.

Jackson, M. O. (2008): *Social and Economic Networks.* Princeton University Press, Pinceton, NJ, USA.

Jackson, M. O., M. Elliot, and B. Golub (2014): "Financial Networks and Contagion," *American Economic Review*, 104(9).

Jackson, M. O., T. Rodriguez-Barraquer, and X. Tan (2012): "Social Capital and Social Quilts: Network Patterns of Favor Exchange," *American Economic Review*, 102(5), 1857–1897.

Jackson, M. O., and B. Rogers (2005): "The Economics of Small Worlds," *Journal of the European Economic Association*, 3(2).

Jackson, M. O., and A. Wolinsky (1996): "A Strategic Model of Social and Economic Networks," *Journal of Economic Theory*, 71, 44–74.

Kalai, E., A. Postlewaite, and J. Roberts (1978): "Barriers to Trade and Disadvantageous Middlemen: Nonmonotonicity of the Core," *Journal of Economic Theory*, 19, 200–209.

Krackhardt, D. (1987): "Cognitive Social Structures," *Social Networks*, 9, 109–134.

Kranton, R., and D. Minehart (2001): "A Theory of Buyer-Seller Networks," *American Economic Review*, 91, 301–332.

Myerson, R. B. (1991): *Game Theory: Analysis of Conflict*. Harvard University Press, Cambridge, MA.

Polanski, A. (2007): "Bilateral bargaining in networks," *Journal of Economic Theory*, 134(1), 557–565.

Rubinstein, A., and A. Wolinsky (1987): "Middlemen," *Quarterly Journal of Economics*, 102, 581–593.

Schumpeter, J. (1942): *Capitalism, Socialism and Democracy*. HarperPerennial, New York, NY, 3rd edn.

Sims, O., and R. P. Gilles (2014): "Critical Nodes in Directed Networks," *ArXiv e-prints*, (1401.0655v2).

Spiro, E. S., R. M. Acton, and C. T. Butts (2013): "Extended Structures of Mediation: Re–Examining Brokerage in Dynamic Networks," *Social Networks*, 35(1), 130–143.

Vitali, S., J. B. Glattfelder, and S. Battiston (2011): "The Network of Global Corporate COntrol," *PLoS ONE*, 6(10), e25995.

A Appendix: Modifications of the brokerage index τ

A.1 Distance-based brokerage

We extend our discussion of the criticality of nodes with a measure that combines middleman power with node proximity. Consider a directed network D on N and $i, j, h \in N$ with $h \in \mathcal{M}(D)$. The power of middleman h could be less effective due to the shortest walk from i to j. Consider an amended brokerage score to capture this effect given by:

$$\Delta_{ij}(h) = \frac{1}{\delta_{ih}} \cdot \frac{1}{\delta_{hj}},$$

where $\delta_{ij} \in \arg\min \left\{ \#W_{ij}(D) \mid W_{ij}(D) \in \mathcal{W}_{ij}(D) \right\}$ is the shortest walk from i to j in the network D. Here, nodes closer to h provide a greater brokerage power to node h than those at larger distances. Indeed, h receives maximal brokerage power if $i \in p_h(D)$ and $j \in s_h(D)$.

Definition A.1 *The **distance–based brokerage** for $h \in M(D)$ is defined as*

$$\nu_h^*(D) = \sum_{i,j \in N : h \in M_{ij}(D)} \Delta_{ij}(h).$$

It is particularly beneficial to use this modified measure to assess costly trade in a network where costs are constant across arcs, or the diffusion of information that can degrade as it is being passed through a network. This assumption of information degradation and even complete truncation over a certain distance has been widely used in literature regarding social networks (Jackson and Rogers, 2005; Jackson, 2008).

A.2 Weighted brokerage

The analysis above looks at binary directed networks only. However, many networks can be weighted in terms of the intensity of the relationship from i to j. We denote the weight on an arc (i, j) by ξ_{ij}, where $0 < \xi_{ij} \leqslant 1$ if there exists an arc and 0 otherwise. The intuition is that the intensity of the relationship from i to j increases as ξ_{ij} converges to 1.

Given a weighted directed network all (i,j)–walks can be weighted. The weight of an (i,j)-walk is given by the product of the weights of all arcs in the given walk from node i to node j. The strongest walk from node i to node j is therefore considered to be the (i,j)-walk with the largest weight. Note that this does not mean that it is the shortest walk from i to j; but instead is equal to the walk with the largest weights. More formal definitions are given in Definition A.2.

Definition A.2 *Let D be a weighted directed network on node set N where $i, j \in N$ and $\mathcal{W}_{ij} \neq \emptyset$ such that $i \neq j$.*

(a) *The **walk weight** of some walk $W_{ij} \in \mathcal{W}_{ij}(D)$, denoted by $\Xi_{W_{ij}}$, is defined by:*

$$\Xi_{W_{ij}}(D) = \xi_{i_1 i_{i+1}} + \ldots + \xi_{i_{m-1} i_m},$$

where $m \geqslant 3$, $i_1 = i$, $i_m = j$, and $i_k, i_{k+1} \in D$ for every $k = 1, \ldots, m-1$.

(b) *The **strongest (i,j)-walk** for two distinct nodes, i and j, is given as*

$$\hat{\Xi}_{W_{ij}}(D) = \arg\max \left\{ \Xi_{W_{ij}}(D) \,\middle|\, W_{ij}(D) \in \mathcal{W}_{ij}(D) \right\}.$$

(c) *The **weighted brokerage** of node $h \in N$ is given as*

$$O_h(D) = \sum_{i,j \in N : h \in M_{ij}(D)} \hat{\Xi}_{W_{ij}}(D).$$

Note that the value for the weighted brokerage must be less than or equal to the brokerage of the same node set. Furthermore, the weighted brokerage measure only takes into consideration the *strongest* walks only; a more conservative measure could be created in which only the *weakest*,i.e. minimal weighted, walks are considered. Alternatively, an average measure could be constructed that takes the average weights of all walks.

B Appendix: Proofs of the main results

Proof of Theorem 2.3

Let $B \subset N$ be some arbitrary node set. For B to be a middleman or block between two nodes $i, j \in N$, where $i \neq j$, it must be that $B \cap W_{ij}(D) \neq \varnothing \,\forall\, W_{ij}(D) \in \mathcal{W}_{ij}(D)$ where $i, j \notin B$.

A fundamental requirement for there to exist a middleman or block is that $W_{ij} \setminus \{i, j\} \neq \varnothing$. First, we note that the minimum value for $\#W_{ij}(D) = 2$, given that $i \neq j$; this is the case where $j \in s_i(D)$ and thus $i \in p_j(D)$. Next, $\#W_{ii}(D) = 1$ if and only if $W_{ii}(D) = \{i\}$, which infers a *self–link*. Finally, $\#W_{ij}(D) = 0$ if and only if $W_{ij}(D) = \varnothing$. Let $W_{ij}^{min} \in \min \left\{ W_{ij}(D) \mid W_{ij}(D) \in \mathcal{W}_{ij}(D) \right\}$ be the geodesic walk from some node i to j. If $\#W_{ij}^{min}(D) \leqslant 2 \,\forall\, i, j \in N$ then it is clear that $W_{ij}^{min} \setminus \{i, j\} = \varnothing \,\forall\, i, j \in N$, meaning that both middlemen and blocks are unable to emerge.

If there exists some $i, j \in N$ where $i \neq j$ such that $\min \left\{ \#W_{ij}(D) \mid W_{ij}(D) \in \mathcal{W}_{ij}(D) \right\} > 2$ then $\exists\, h \in N$ such that $h \in W_{ij}(D) \setminus \{i, j\}$ where $W_{ij}(D) \in \mathcal{W}_{ij}(D)$. Node h may or may not be a middleman, however we claim that it can be part of a block.

For block B to exist it must hold that $B \cap \mathcal{W}_{ij} \setminus \{i, j\} \neq \varnothing$ for all $W_{ij}(D) \in \mathcal{W}_{ij}(D)$ and $\#B > 1$. If $\min \left\{ \#W_{ij}(D) \mid W_{ij}(D) \in \mathcal{W}_{ij}(D) \right\} \geqslant 3$ then there must exist some node set $B \subseteq N \setminus \{i, j\}$, therefore there must exist either a block or a middleman.

Proof of Theorem 2.7

Proof of Theorem 3.2

Consider a situation where there exists a partition of $\mathcal{A}$, given by $R(\varphi) = \left(S^1, \ldots, S^K \right)$, such that $\exists\, S^k \in R(\varphi)$ where $S^k \notin \mathcal{A}(\tilde{a})$, and $\tilde{a}$ is some SNE.

If $S^k \notin \mathcal{A}(\tilde{a})$ then there must exist some $S^{k'} \in \mathcal{A}$ such that $S^k \cap S^{k'} \neq \varnothing$, $\varphi(S^{k'}) > \varphi(S^k)$, and $S^{k'} \in \mathcal{A}(\tilde{a})$. If this is true then it must be that $S^{k'} \in R(\varphi)$ and $S^k \notin R(\varphi)$ due to the conditions of the algorithm for creating $R(\varphi)$. Specifically, if $\varphi(S^{k'}) > \varphi(S^k)$

and $\nexists S \in \mathcal{A}$ such that $S^k \cap S^{k'} \neq \emptyset$, $\varphi(S^{k'}) > \varphi(S^k)$, and $S^{k'} \in R(\varphi)$ then $S^{k'} \in R(\varphi)$. Therefore $S^k \notin R(\varphi)$. Ultimately, $R(\varphi)$ where $S^k \in R(\varphi)$ could not have been a SNE.

It may be the case that there can exist some $S^{k''} \in \mathcal{A}$ such that $S^k \cap S^{k''} \neq \emptyset$, $\varphi(S^k) = \varphi(S^{k''})$, and there exists no other node set such that there is a non–empty intersection between it and either S^k and $S^{k''}$ and it has a higher maximum individual payoff which is also in the SNE. Then it will be the case that either S^k or $S^{k''}$ can be in a SNE but not simultaneously. Indeed, $R(\varphi)$ must contain all actions that are in SNE.

Proof of Theorem 3.6

If: The formation of blocks requires consent from all of its members, i.e., multiple players. The exploitation of a nodes own position does not require consent from multiple agents; rather, if the agent gains a higher individual payoff from exploiting her own position than participating in a block then she will do so. Formally, if $\exists\, i \in B : \sigma(i) > \varphi(B)$, where $\sigma(i) = \varphi(i)$, then $\pi_i(a) > \pi_i(a')$ where $a_i = i$, $a'_i = B$, and $a_{-i} = a'_{-i}$ for any a_{-i}. Indeed, regardless of the actions of other players, i will always be better off pursuing the exploitation of her own individual position rather than operating in a block.

If $\nexists\, i \in B : \sigma(i) > \varphi(B)$ then player i will have no incentive to exploit her own position unless $\exists j \in B$ such that $a_j \neq B$ and $c > 0$. If $\nexists j \in B$ such that $a_j \neq B$ then $a_i^\star = B$.

Only if: We must prove that some action $B \in \mathcal{A}(a^\star)$ must be in some NE if $\nexists\, i \in B : \sigma(i) > \varphi(B)$. Indeed, action B will be in a NE if all players consent to its formation, and therefore no player has any incentive to deviate strategies. A player will only deviate from the formation of some action B if they receive a higher individual payoff from doing so. This will occur if they receive a larger payoff from operating individually (as noted above) or operating in another block. Since the formation of a block requires consent from all of its members, and the best response to at least one other member of the block not pursuing the block is to also not pursue the block, then the formation of an alternative block that derives a higher individual payoff will not be viable under NE conditions.

Specifically, we note that the NE here is analogous to *link deletion proof*, but not *link addition proof* (Jackson and Wolinsky, 1996).

Proof of Theorem 4.3

This proof follows is a continuation and application of the proof derived for Theorem 3.2. Let $B \subset N$ be a redundant block, such that $\exists\, B' \subset B$ where $\mathcal{Z}_{B'}(D) \supseteq \mathcal{Z}_B(D)$. Since $B' \subset B$ it must be that $\varphi_{B'} > \varphi_B$ and $B' \cap B \neq \emptyset$. $B' \in R(\varphi)$ if $\nexists\, B'' \in R(\varphi)$ such that $B'' \cap B' \neq \emptyset$. If there exists $B'' \in R(\varphi)$ then $B' \notin R(\varphi)$, but also $B \notin R(\varphi)$ since $B' \subset B$.

Under SNE conditions a non–redundant node set will always be selected over the equivalent redundant node set since the cardinality is higher in the redundant node set than the non–redundant node set.

Proof of Theorem 4.5

Following from Theorem 3.6 if $\varphi(i) > \varphi(B) \, \forall \, B \in \mathcal{B}_i(D)$ then player $i \in N$ will always only wish to exploit her own position in the network. We can extend this finding to all $B \in \mathcal{B}_i(D)$.

In general we show that $\varphi(i) \geqslant \varphi(B)$, which is the condition required for i to exploit her position as opposed to participating in $B \in \mathcal{B}_i(D)$.

$$
\begin{aligned}
\varphi(i) &\geqslant \varphi(B) \\
\sigma(i) &\geqslant \frac{\sigma(B)}{\#B}(\#B - 1)c \\
\#B(\#Z_i(D)) + \#B(\#B - 1)c &\geqslant \sigma(B) \\
\#B\,[(\#\mathcal{Z}_i(D)) + (\#B - 1)c] &\geqslant \#\mathcal{Z}_i(D) + \#K_i \\
\#B(\#\mathcal{Z}_i(D)) - \mathcal{Z}_i(D) + \#B(\#B - 1)c &\geqslant \#K_i \\
(\#B - 1)\#\mathcal{Z}_i(D) + (\#B - 1)\#Bc &\geqslant \#K_i \\
(\#B - 1)(\#Bc + \#\mathcal{Z}_i(D)) &\geqslant \#K_i
\end{aligned}
$$

We clarify that $\#\mathcal{Z}_B(D) = \#\mathcal{Z}_i(D) + \#K_i$, since $\mathcal{Z}_i(D) \cap K_i = \emptyset$ and also note that $\sigma(B) \equiv \#\mathcal{Z}_i(D) + \#K_i$ due to the initial condition $\mathcal{Z}_B(D) = \mathcal{Z}_i(D) \cup K_i$.

Proof of Theorem 4.6

Proof of (a) Consider the case where $\varphi(i) < \varphi(B)$. This implies that $\mathcal{Z}_i(D) < \frac{\mathcal{Z}_B(D)}{\#B} - (\#B - 1)c$. Given that $\mathcal{Z}_B(D) = \mathcal{Z}_{B'}(D) + \mathcal{Z}_i(D)$ since i is uncontested, we claim that block B is always unstable in SNE since $\varphi(B) < \varphi(B')$ where $B' = B \setminus \{i\}$.

The condition to satisfy $\varphi(B) < \varphi(B')$ can be written as:

$$
\frac{\mathcal{Z}_B(D)}{\#B} - (\#B - 1)c < \frac{\mathcal{Z}_B(D) - \mathcal{Z}_i(D)}{\#B - 1} - (\#B - 2)c. \tag{20}
$$

By re–arranging we note that $\varphi(B) < \varphi(B')$ is satisfied when $\mathcal{Z}_i(D) > \frac{\mathcal{Z}_B(D)}{\#B} + 2c\#B - 3c$. If we let $c = 0$, which is the least restrictive assumption, then condition shown in Equation 20 is satisfied when $\mathcal{Z}_i(D) > \frac{\mathcal{Z}_B(D)}{\#B}$. However, this is infeasible since we are specifically considering the case where $\varphi(i) < \varphi(B)$.

We make a note that since $\#B > 1$ for a block to exist, as c increases above 0, the value $\frac{\mathcal{Z}_B(D)}{\#B} + 2c\#B - 3c$ will continue to increase therefore making the condition shown in Equation 20 impossible to satisfy given the initial restriction that $\varphi(i) < \varphi(B)$. Under this case, the players that comprise the block, B, will wish to remove i from the block since her marginal contribution to the block is too low.

Proof of (b) Consider the situation where $\varphi(i) > \varphi(B)$. From Theorem 3.6 we note that a middleman will have an incentive to exploit their own position in the network as opposed to participate in the block if the payoff from exploiting her own position is larger than the individual payoff for participating in the block. Due to the property that all uncontested nodes are middlemen, the intuition from Theorem 3.6 is extended to this. Therefore, the block will also not be stable in a SNE due to i's incentive to deviate.